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SPECTRA OF CONVOLUTION OPERATORS ON L^p AND RINGS OF FACTOR-SEQUENCES

By G. L. KRABBE (*Purdue University, U.S.A.*)

[Received 9 June 1956]

1. Introduction

SUPPOSE $1 < p < \infty$. A sequence (a_n) $(n \in \Omega)$ is said to be of *type* (L^p, L^p),† if $(a_n c_n)$ $(n \in \Omega)$ are the Fourier coefficients of a function g in L^p whenever (c_n) $(n \in \Omega)$ are the Fourier coefficients of some function f in L^p. The mapping $f \to g$ is then a bounded linear operator $T_{a,p}$; the point-spectrum of $T_{a,p}$ being the set $\{a_n : n \in \Omega\}$ (i.e. the range of a), it follows that

$$\sigma(T_{a,p}) \supseteq \varpi(a),$$

where $\sigma(T_{a,p})$ denotes the spectrum of $T_{a,p}$, and where $\varpi(a)$ denotes the closure of the set $\{a_n : n \in \Omega\}$.

Suppose that a is any bounded sequence. It is easy to prove:

THEOREM 1. *If $p = 2$, then the sequence a is of type (L^p, L^p) and $\sigma(T_{a,p}) = \varpi(a)$.*

It will be shown in this paper that Theorem 1 holds for all values of p when a is suitably restricted. More specifically, I shall define the 'σ-variation' $V_\sigma(a)$ of the sequence a [see § 7], and verify that, if $V_\sigma(a) < \infty$, then the conclusion of Theorem 1 is valid for any p in $(1, \infty)$; moreover, there exists a number $k_p > 0$ such that

$$\text{(i)} \quad \sup_{n \in \Omega} |a_n| = \lim_{\nu \to \infty} \|(T_{a,p})^\nu\|_p^{1/\nu} \leqslant \|T_{a,p}\|_p \leqslant k_p\Big(\sup_{n \in \Omega} |a_n| + V_\sigma(a)\Big).$$

The set $\mathscr{R}_\sigma$ of all bounded sequences a with $V_\sigma(a) < \infty$ forms a Banach algebra, and the set of all sequences of bounded variation‡ forms a proper sub-algebra of $\mathscr{R}_\sigma$. We also have, for $\mathscr{R} = \mathscr{R}_\sigma$,

(ii) *if f is an analytic function on the set $\varpi(a)$, then the sequence $(f(a_n))$ $(n \in \Omega)$ is in $\mathscr{R}$ when $a = (a_n)$ $(n \in \Omega)$ is in $\mathscr{R}$.*

† See (**3**) 17.2. Such sequences are also called 'factor-sequences' or 'multiplicators'.

‡ See (**5**) 230. A sequence a is of *bounded variation* if $\sum |a_n - a_{n+1}| < \infty$ $(n = -\infty, \ldots, \infty)$.

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2. Application to certain integral operators

First, suppose that $A \in L^1$, and denote by A_{*p} the mapping $f \to A * f$ of L^p into itself; here $A * f$ is the resultant† of A and f. It is easily seen that $A_{*p} = T_{a,p}$, where a is the sequence of Fourier coefficients ΦA of A. Accordingly, the point-spectrum of A_{*p} is the range of ΦA, and the spectrum of any such convolution operator has therefore an accumulation point at the origin. Suppose that A is restricted to the space of absolutely continuous functions with period 2π; then ΦA is in $\mathscr{R}_\sigma$. Consequently, (i)–(ii) hold, and

$$\sigma(A_{*p}) = \varpi(\Phi A).$$

[This is the dual of a previous result (**6**), which concerns the mapping $a_{*p}: c \to a * c$ of the sequence space l_p into itself; if $a = \Phi A$, then $\sigma(a_{*p}) = \varpi(A)$.]

Next, integral operators of another kind are exemplified by the transformation T which maps any function f of L^p onto the function g defined by

$$2\pi g(\theta) = \text{principal value of } \int_{-\pi}^{\pi} f(\lambda) \cot \tfrac{1}{2}(\theta - \lambda)\, d\lambda. \tag{1}$$

Let a be the sequence $(-i \operatorname{sgn} n)$ $(n \in \Omega)$ [see § 7.2]. If (c_n) $(n \in \Omega)$ are the Fourier coefficients of f, it is known [(**10**) 490] that $(a_n c_n)$ $(n \in \Omega)$ are the Fourier coefficients of g. Therefore $T = T_{a,p}$; but a is a sequence of bounded variation, and hence $a \in \mathscr{R}_\sigma$. Consequently, the assertions made in § 1 yield immediately that $\sigma(T) = \{-i, 0, i\}$ is the point-spectrum of T, and that $1 \leqslant \|T\|_p$. This procedure can apply to the more general operators T studied by Calderón and Zygmund since to each such T corresponds a sequence a with $T = T_{a,p}$ [see (**1**) 259].

3. Preliminaries

The following three sets are given and will remain fixed until § 7:

(*a*) a compact interval $[\alpha, \beta] \subseteq (-\infty, \infty)$,

(*b*) a subset Ω of $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$,

(*c*) an orthonormal‡ set $\{\Phi_n : n \in \Omega\} \subseteq L^\infty[\alpha, \beta]$ of functions complete in each of the spaces $L^r[\alpha, \beta]$ $(1 \leqslant r \leqslant \infty)$.

The letter p will consistently denote a number satisfying $1 \leqslant p \leqslant \infty$. As usual, $L^\infty[\alpha, \beta]$ consists of essentially-bounded summable functions; we shall write $L^p = L^p[\alpha, \beta]$. If $x \in L^1$, then Φx will denote the sequence $([\Phi x]_n)$ $(n \in \Omega)$ defined by

$$[\Phi x]_n = \frac{1}{\sqrt{(2\pi)}} \int_\alpha^\beta x(\theta) \overline{\Phi_n(\theta)}\, d\theta \quad (n \in \Omega).$$

† Or 'convolution'; see § 8.

‡ The orthonormality is on $[\alpha, \beta]$.

The hypotheses imply that $[\Phi\Phi_n]_\nu = (2\pi)^{-\frac{1}{2}}\delta_\nu^n$ (the Kronecker delta)† and also that $x \to \Phi x$ is a one-to-one mapping of L^p onto

$$\Phi(L^p) = \{\Phi x : x \in L^p\}.$$

3.1. *Notation.* A mapping $z \to y$ defined for all z in a set Z will be written $\{z \in Z \to y\}$; thus, for a function f defined on Z, we have

$$f = \{z \in Z \to f(z)\}.$$

For example, $$(f(a_n))\,(n \in \Omega) = \{n \in \Omega \to f(a_n)\}.$$

3.2. DEFINITIONS. If $a = (a_n)$ $(n \in \Omega)$ and $b = (b_n)$ $(n \in \Omega)$, then $a \circ b = (a_n b_n)$ $(n \in \Omega)$. The set $\mathscr{A}_p$ of all sequences of *type* (L^p, L^p) consists of sequences a such that $a \circ (\Phi x)$ is in $\Phi(L^p)$ for all x in L^p. If $a \in \mathscr{A}_p$ and $x \in L^p$, then $T_{a,p}x$ denotes the unique y in L^p such that $a \circ (\Phi x) = \Phi y$. If $a \in \mathscr{A}_p$, then $T_{a,p}$ denotes the mapping

$$\{x \in L^p \to T_{a,p}x\}.$$

3.3. *Remarks.* Suppose that $a \in \mathscr{A}_p$. From the definition it follows that

(*d*) *if* $x \in L^p$, *then* $a \circ (\Phi x) = \Phi(T_{a,p}x)$ *and* $T_{a,p}x \in L^p$.

The range of a is the set $\{a_n : n \in \Omega\}$; it is easily checked that

(*e*) *the point-spectrum of* $T_{a,p}$ *is the range of* a.

3.4. THEOREM. *The set* $\mathscr{A}_p$ *forms a commutative algebra, and the mapping* $\{a \in \mathscr{A}_p \to T_{a,p}\}$ *is an isomorphism into the space* $\mathfrak{E}_p$ *of all bounded linear transformations of* L^p *into itself.*

Proof. Suppose that $a \in \mathscr{A}_p$. The fact that $T_{a,p} \in \mathfrak{E}_p$ is known [(**5**) 222]. The only essential point in the remainder of this proof is to show that $T_{a,p}x = \mathbf{0}$ $(x \in L^p)$ implies $a = \mathbf{0}$; supposing that $n \in \Omega$ and $x = \Phi_n$, we then have $a \circ (\Phi\Phi_n) = \Phi(T_{a,p}x) = 0$, whence

$$a_n[\Phi\Phi_n]_n = (2\pi)^{-\frac{1}{2}}a_n = 0$$

and $a_n = 0$.

4. On algebras of bounded sequences

Let $\mathscr{B}$ denote the commutative algebra of all bounded sequences (multiplication being defined as in § 3.2); note that the multiplicative unit of $\mathscr{B}$ is the sequence $I^0 = (1)$ $(n \in \Omega)$. If $a \in \mathscr{B}$, then a is a sequence (a_n) $(n \in \Omega)$, and we write

$$\|a\|_\infty = \sup\{|a_n| : n \in \Omega\},$$

while $\varpi(a)$ denotes the closure of the range of a [see § 3.3].

† The *raison d'être* for the factor $(2\pi)^{-\frac{1}{2}}$ will not appear until § 7.

4.1. DEFINITIONS. If $\mathscr{A}$ is a sub-algebra of $\mathscr{B}$, I shall write $\mathscr{R} \prec \mathscr{A}$ to indicate that $\mathscr{R}$ is a sub-algebra of $\mathscr{A}$ containing I^0. If $\mathscr{R} \prec \mathscr{B}$, then $\mathscr{R}^\sharp$ is the set of all elements of $\mathscr{R}$ having inverses in $\mathscr{R}$; if, also, $a \in \mathscr{R}$, then $\sigma(a, \mathscr{R})$ will denote the set of all complex λ such that $(\lambda I^0 - a) \notin \mathscr{R}^\sharp$.

4.2 THEOREM. *We have* $\mathscr{A}_p \prec \mathscr{B}$. *If* $\mathscr{R} \prec \mathscr{A}_p$ *and* $\mathscr{R} \cap \mathscr{B}^\sharp \subseteq \mathscr{R}^\sharp$, *then* $a \in \mathscr{R}$ *implies*

$$\text{(iii)} \qquad \varpi(a) = \sigma(T_{a,p}) = \sigma(a, \mathscr{R}),$$

$$\text{(i')} \qquad \|a\|_\infty = \lim_{\nu \to \infty} \|(T_{a,p})^\nu\|_p^{1/\nu} \leqslant \|T_{a,p}\|_p.$$

4.3. *Remarks.* As usual,

$$\|T_{a,p}\|_p = \sup\{\|T_{a,p}\, x\|_p \colon \|x\|_p \leqslant 1\}.$$

Suppose that r is some number between p and $p' = p/(p-1)$; the Riesz convexity theorem† now implies that‡

$$\|T_{a,r}\|_r \leqslant \|T_{a,p}\|_p.$$

Suppose that $a \in \mathscr{A}_1$; it follows that

$$f_\nu(p) = \|(T_{a,p})^\nu\|_p^{1/\nu}$$

is a non-increasing function of p in the closed interval $[1, 2]$ and is non-decreasing in $[2, \infty]$. This may render more striking the fact (i') that

$$\|a\|_\infty = \lim_{\nu \to \infty} f_\nu(p),$$

regardless of the value of p in $[1, \infty]$. In view of the above, it is readily derived from certain results due to Orlicz§ that

$$\mathscr{A}_1 \cup \mathscr{A}_\infty \subseteq \mathscr{A}_p = \mathscr{A}_{p'} \subseteq \mathscr{A}_r = \mathscr{A}_{r'} \subseteq \mathscr{A}_2 = \mathscr{B}. \tag{2}$$

To ensure that $\mathscr{A}_1 = \mathscr{A}_\infty$ it suffices to suppose that $\{\Phi_n \colon n \in \Omega\}$ is complete in L^1 and that the Φ_n are bounded functions.‖

It is clear that $0 \notin \varpi(a)$ if and only if there is an $\epsilon > 0$ with $a_n \geqslant \epsilon$ for $n \in \Omega$. If $a^{-1} = ((a_n)^{-1})$ $(n \in \Omega)$, then

$$\mathscr{R}^\sharp = \{c \in R \colon c^{-1} \in \mathscr{R}\},$$

and we can paraphrase (iii) as follows: *the inverse* $T_{a,p}^{-1}$ *of* $T_{a,p}$ *exists if and only if* $0 \notin \varpi(a)$; *if* $T_{a,p}^{-1}$ *exists, then* $T_{a,p}^{-1} = T_{a^{-1},p}$.

4.4. *Proof of* § 4.2. From § 4.3 and § 3.4 it is seen that $\mathscr{A}_p \prec \mathscr{B}$. In order to establish (iii), we first prove that

$$\varpi(a) \subseteq \sigma(T_{a,p}) \subseteq \sigma(a, \mathscr{R}). \tag{3}$$

† See (**10**) 481 and (**13**) 198.

‡ For a careful derivation of this fact, see (**4**), Lemma 2.4.

§ (**5**) 222–4. See also (**4**) 337 and (**9**) 22.

‖ See (**5**) 224–5; we thus relax the unnecessarily restrictive conditions of (**4**) and (**9**), loc. cit.

To that effect, note that $\sigma(T_{a,p})$ includes the point-spectrum of $T_{a,p}$ which, by § 3.3 (e), includes the range of a; the closed set $\sigma(T_{a,p})$ therefore includes $\varpi(a)$. Suppose that $\lambda \in \sigma(T_{a,p})$; the proof of (3) is completed by showing that $\lambda \in \sigma(a, \mathscr{R})$. By way of contradiction, assume that

$$c = (\lambda I^0 - a) \in \mathscr{R}^z.$$

Then $$b \circ (\lambda I^0 - a) = I^0$$

for $b = c^{-1}$ in $\mathscr{R}$, whence

$$Q(\lambda I - T_{a,p}) = I$$

for $Q = T_{b,p}$ in $\mathfrak{E}_p$, by § 3.4, and the contradiction $\lambda \notin \sigma(T_{a,p})$ now follows.

Next, I shall show that $\sigma(a, \mathscr{R}) \subseteq \varpi(a)$: this will establish (iii), in view of (3). Assume that $\lambda \in \sigma(a, \mathscr{R})$; if $\lambda \notin \varpi(a)$, then $|\lambda - a_n| \geqslant \epsilon$ for all n and some $\epsilon > 0$. Note that

$$c = (\lambda I^0 - a) \in \mathscr{R}, \qquad |c_n| = |\lambda - a_n| \geqslant \epsilon,$$

whence $c^{-1} \in \mathscr{B}$, and therefore $c \in \mathscr{B}^z$. But then $c \in \mathscr{R} \cap \mathscr{B}^z$, which implies the contradiction $c = (\lambda I^0 - a) \in \mathscr{R}^z$ of our assumption

$$\lambda \in \sigma(a, \mathscr{R}).$$

In order to prove (i'), set

$$d[E] = \sup\{|\lambda| : \lambda \in E\},$$

and note that

$$\|a\|_\infty = d[\varpi(a)] = d[\sigma(T_{a,p})] = \lim \|(T_{a,p})^\nu\|_p^{1/\nu} \leqslant \|T_{a,p}\|_p.$$

The second equality follows from (iii), and the third is well known [(11) 414].

5. On Banach-algebras

It will now be shown that, if a Banach algebra $\mathscr{R}$ satisfies the hypotheses of § 4.2, then the property (ii) of § 1 holds, and the isomorphism $a \to T_{a,p}$ is a continuous representation of $\mathscr{R}$. We say that $\mathscr{R}$ is *finer than* $\mathscr{B}$ if the norm $\|a\|_\infty$ of any a in $\mathscr{B}$ never exceeds the norm $\|a\|$ of a in $\mathscr{R}$. If Σ is a closed set, we say that a function is *analytic on* Σ whenever it is defined and holomorphic on an open set containing Σ; note that, if $a \in \mathscr{R}$, then the spectrum of a in $\mathscr{R}$ is the closed set $\sigma(a, \mathscr{R})$.

5.1. Lemma. *If $\mathscr{R}$ is a Banach algebra with $\mathscr{R} \prec \mathscr{B}$, then $\mathscr{R}$ is finer than $\mathscr{B}$. If f is analytic on $\sigma(a, \mathscr{R})$, then $(f(a_n))$ $(n \in \Omega)$ is in $\mathscr{R}$ when $a \in \mathscr{R}$.*

Proof. Suppose that $n \in \Omega$. We first show that the mapping ψ_n defined

by $\{a \in R \to a_n\}$ belongs to the space $\mathscr{R}^\Delta$ of all homomorphisms onto the complex plane P. This results from

$$\psi_n(a \circ b) = [a \circ b]_n = \psi_n(a)\psi_n(b), \qquad \lambda = \psi_n(\lambda I^0)$$

with $\lambda I^0 \in \mathscr{R}$ for all λ in P. Therefore $\psi_n \in \mathscr{R}^\Delta$; since this implies that $\|\psi_n\| \leqslant 1$ [(7) 23A], we have thus also proved that

$$|\psi_n(a)| = |a_n| \leqslant \|a\|,$$

so that $\mathscr{R}$ is finer than $\mathscr{B}$. On the other hand, from $\psi_n \in \mathscr{R}^\Delta$ and (7) 23A, it follows that there exists a maximal ideal M of $\mathscr{R}$ such that $\psi_n(c) = \hat{c}(M)$ for all c in $\mathscr{R}$ [(7) 23B]. If $a \in \mathscr{R}$, we can infer from (7) 24D that there exists a member $y = (y_n)$ $(n \in \Omega)$ of $\mathscr{R}$ such that $\hat{y}(M) = f(\hat{a}(M))$; therefore $\psi_n(y) = f(\psi_n(a))$, whence $y_n = f(a_n)$. Since n is arbitrary, we can conclude that

$$(f(a_n))\ (n \in \Omega) = y \in \mathscr{R}.$$

5.2. Lemma. *If $\mathscr{R} \subseteq \mathscr{A}_p$ and if $\mathscr{R}$ is a Banach space finer than $\mathscr{B}$, then the mapping $\{a \in \mathscr{R} \to T_{a,p}\}$ is a bounded linear transformation.*

Proof. Let $h(a) = T_{a,p}$. It will suffice to show that h is closed since h is linear by § 3.4 and (3) 30. To that effect, suppose that

$$a^l \in \mathscr{R}, \qquad c \in \mathscr{R}, \qquad \lim\|a^l - c\| = 0 \quad (l \to \infty \text{ throughout}),$$

$$\lim\|h(a^l) - Q\|_p = 0.$$

We complete the proof by establishing $h(c) = Q$. Set $x \in L^p$ and note from § 3.3 that

$$\Phi(h(b)x) = b \circ (\Phi x) \quad (b \in \mathscr{A}_p). \tag{4}$$

Since $\mathscr{R}$ is finer than $\mathscr{B}$, we have

$$\lim[a^l \circ (\Phi x)]_n = [c \circ (\Phi x)]_n,$$

so that, by (4),

$$\lim[\Phi(h(a^l)x)]_n = [c \circ (\Phi x)]_n. \tag{5}$$

On the other hand, our hypotheses imply that

$$\lim\|h(a^l)x - Qx\|_p = 0,$$

which implies

$$\lim[\Phi(h(a^l)x)]_n = [\Phi(Qx)]_n;$$

by comparing this with (5), we obtain $c \circ (\Phi x) = \Phi(Qx)$; but (4) then shows that $\Phi(h(c)) = \Phi(Qx)$, whence $h(c)x = Qx$ and the conclusion $h(c) = Q$.

5.3. THEOREM. *Let $\mathscr{R}$ be a Banach algebra. If $\mathscr{R} \prec \mathscr{A}_p$ and*

$$\mathscr{R} \cap \mathscr{B}^\sharp \subseteq \mathscr{R}^\sharp,$$

then there exists a number k_p (> 0) such that, for any a in $\mathscr{R}$,

(i) $\|a\|_\infty = \lim_{\nu\to\infty} \|(T_{a,p})^\nu\|_p^{1/\nu} \leqslant \|T_{a,p}\|_p \leqslant k_p\|a\|$,

(ii) *if f is an analytic function on the set $\varpi(a)$, then the sequence $(f(a_n))$ $(n \in \Omega)$ is in $\mathscr{R}$,*

(iii) $\sigma(T_{a,p}) = \varpi(a) = \sigma(a, \mathscr{R})$.

Proof. Since $\mathscr{A}_p \prec \mathscr{B}$, then $\mathscr{R} \prec \mathscr{B}$, and (i), (ii), (iii) follow from § 4.2, § 5.1, and § 5.2.

6. A special Banach algebra

In the case $p = 2$, it is seen from § 5.3 that (i), (ii), (iii) are satisfied for any a in $\mathscr{R}$ when $\mathscr{R} = \mathscr{A}_2$ since $\mathscr{A}_2$ is the Banach algebra $\mathscr{B}$ [see § 4.3]. We now lay the groundwork for a sub-algebra of $\mathscr{A}_2$ such that (i), (ii), (iii) hold for any p in $(1, \infty)$.

6.1. DEFINITIONS. Let s be a sequence (s_ι) $(\iota \in \Omega)$ of finite† subsets of Ω; then $\mathscr{R}_s$ will denote the set $\{a \in \mathscr{B}: V_s(a) < \infty\}$, where

$$V_s(a) = \sup_{\iota\geqslant 0} \sum_{n\in s_\iota} |[\Delta a]_n|, \qquad [\Delta a]_n = a_n - a_{n+1}.$$

6.2. THEOREM. *The norm $\|a\| = \|a\|_\infty + V_s(a)$ makes $\mathscr{R}_s$ into a Banach algebra, and $\mathscr{R}_s \prec \mathscr{B}$.*

Proof. To show the completeness of $\mathscr{R}_s$, suppose that

$$a^n \in \mathscr{R}_s, \qquad \lim\|a^n - a^m\| = 0 \quad (n, m \to \infty).$$

We then have $\qquad a^n \in \mathscr{B}, \qquad \lim\|a^n - a^m\|_\infty = 0,$

and there exists some b in $\mathscr{B}$ with $\lim\|a^m - b\|_\infty = 0$ since $\mathscr{B}$ is complete under the norm $\|a\|_\infty$. On the other hand, for any ϵ (> 0) and all m, n sufficiently large, we have

$$V_s(a^n - a^m) < \epsilon;$$

letting $m \to \infty$, we obtain

$$V_s(a^n - b) \leqslant \epsilon,$$

since $\lim(a^m)_\nu = b_\nu$, and therefore $\lim V_s(a^n - b) = 0$ for $n \to \infty$, whence the completeness of $\mathscr{R}_s$. The remaining conclusions are mostly trivial, and I shall merely show that

$$\|c\| \leqslant \|a\|\,\|b\|,$$

† In other words, s_ι is a finite subset of Ω when $\iota \in \Omega$.

where $c = a \circ b$. Since

$$\|c\|_\infty \leqslant \|a\|_\infty \|b\|_\infty,$$

it suffices to establish that

$$V_s(c) \leqslant \|b\|_\infty V_s(a) + \|a\|_\infty V_s(b) + V_s(a)V_s(b).$$

But this follows directly from the easily verifiable relation

$$[\Delta c]_n = b_n[\Delta a]_n + a_{n+1}[\Delta b]_n.$$

6.3. LEMMA. *If s is so chosen that $\mathscr{R}_s \subseteq \mathscr{A}_p$, then the properties* (i), (ii), (iii) *are satisfied for any a in $\mathscr{R} = \mathscr{R}_s$.*

Proof. By § 5.3 and § 6.2, it will suffice to establish that

$$\mathscr{R} \cap \mathscr{B}^\sharp \subseteq \mathscr{R}^\sharp.$$

Suppose $a \in \mathscr{R}$ and $b = a^{-1} \in \mathscr{B}$, i.e. $a \in \mathscr{B}^\sharp$ [see § 4.3]; therefore

$$|[\Delta b]_n| \leqslant (\|b\|_\infty)^2 |[\Delta a]_n|,$$

so that $b \in \mathscr{R}$ and consequently $a \in \mathscr{R}^\sharp$.

7. The σ-variation of a sequence

A theorem of Marcinkiewicz will now enable us to define a sequence σ such that the hypothesis of § 6.3 is verified for $\mathscr{R}_\sigma$ (this is our main objective). The initial statements of § 3 will now be specialized; we restrict ourselves henceforth to the case $\Phi_n(\theta) = (2\pi)^{-\frac{1}{2}} e^{in\theta}$, where

$$\theta \in [\alpha, \beta] = [-\pi, \pi], \qquad n \in \Omega = \{0, \pm 1, \pm 2, \pm 3, \ldots\}.$$

From our definition of Φx it now follows that

$$[\Phi x]_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} x(\theta)\, d\theta \quad (n \in \Omega). \tag{6}$$

I shall use the notation

$$\langle \iota \rangle = \{\nu \in \Omega \colon 2^\iota \leqslant \nu \leqslant 2^{\iota+1}\};$$

the letter σ will denote the sequence (σ_ι) $(\iota \in \Omega)$ defined by

$$\sigma_\iota = \{n \colon n \in \langle \iota \rangle \text{ or } -n-1 \in \langle \iota \rangle\}.$$

In accordance with § 6.1, $\mathscr{R}_\sigma$ is the set of all a in $\mathscr{B}$ such that

$$V_\sigma(a) = \sup\{\beta_\iota \colon \iota \geqslant 0\} < \infty,$$

where

$$\beta_\iota = \sum_{n \in \sigma_\iota} |[\Delta a]_n| = \sum_{\nu \in \langle \iota \rangle} \alpha_\nu; \qquad \alpha_\nu = |[\Delta a]_\nu| + |[\Delta a]_{-\nu-1}|. \tag{7}$$

I call $V_\sigma(a)$ the *σ-variation* of a.

7.1. LEMMA. *If* $a_n = o(|n|^{-1})$, *then* $a \in \mathscr{R}_\sigma$.

Proof. Since our hypothesis states that $\lim |na_n| = 0$ when $|n| \to \infty$, there is a number $N > 0$ such that $|n||a_n| < 1$ when $|n| > N$. If $\nu > N$, then $\alpha_\nu \leqslant 4\nu^{-1}$. Accordingly, if $m \geqslant \mu \geqslant N$, then

$$\sum_{\nu=\mu}^{m} \alpha_\nu < 4\Big(\mu^{-1} + \sum_{\nu=\mu+1}^{m} \nu^{-1}\Big) \leqslant 4\Big(N^{-1} + \int_{\mu}^{m} \theta^{-1}\, d\theta\Big) = 4\Big(N^{-1} + \log\frac{m}{\mu}\Big).$$

If $$\iota > \min\{\iota \geqslant 0 \colon 2^\iota > N\},$$

we can therefore infer from (7) that

$$\beta_\iota < 4(N^{-1} + \log 2^{\iota+1} - \log 2^\iota) = 4(N^{-1} + \log 2).$$

Thus β_ι is bounded for all sufficiently large ι, and consequently $V_\sigma(a) < \infty$. We conclude by noting that $a \in \mathscr{B}$.

7.2. DEFINITIONS. Let $\mathscr{B}^+$ be the set of all bounded sequences u such that $u_n = 0$ for $n < 0$. If $u \in \mathscr{B}^+$, then $\Gamma(u)$ will denote the sequence $(\frac{1}{2}u_{|n|})$ $(n \in \Omega)$. The sequence $j = (-i \operatorname{sgn} n)$ $(n \in \Omega)$ satisfies

$$j_0 = 0, \qquad j_{\pm\nu} = \mp\nu \quad (\nu > 0).$$

7.3. *Remark.* If $1 < p < \infty$ and $c \in \mathscr{A}_p$, then $j \circ c \in \mathscr{A}_p$ [(**10**) 464]; this statement may become more familiar by observing that, if $\sum c_n\, e^{in\theta}$ is the Fourier series of some function f in L^p, then the 'conjugate series' $\sum j_n c_n\, e^{in\theta}$ is the Fourier series of the function g defined in § 2, (1).

7.4. LEMMA. *If* $1 < p < \infty$, $u \in \mathscr{B}^+$, *and* $V_\sigma(u) < \infty$, *then* $\Gamma(u) \in \mathscr{A}_p$.

Proof. From the hypotheses it follows that $\|a\|_\infty < \infty$ and

$$\sum_{\nu=2^\iota}^{-1+2^{\iota+1}} |u_\nu - u_{\nu+1}| \leqslant V_\sigma(u) \quad (\iota \geqslant 0).$$

Suppose that $x \in L^p$ and $a = \Phi x$; from a theorem of Marcinkiewicz,† we have

$$2\Gamma(u) \circ (\Phi x) \in \Phi(L^p),$$

which implies that $2\Gamma(u) \in \mathscr{A}_p$.

7.5. THEOREM. *If* $1 < p < \infty$, *then* $\mathscr{R}_\sigma \subseteq \mathscr{A}_p$, *and the properties* (i), (ii), (iii) *are satisfied for any* a *in* $\mathscr{R} = \mathscr{R}_\sigma$.

Proof. Suppose that $a \in \mathscr{R}_\sigma$, and let u and v be the members of $\mathscr{B}^+$ defined by $v_\nu = a_\nu + a_{-\nu}$ and $w_\nu = i(a_\nu - a_{-\nu})$. It is easily seen that

$$a = \Gamma(v) + j \circ \Gamma(w). \tag{8}$$

The proof will be accomplished by showing that

(*f*) *when* $u = v$ *or when* $u = w$, *then* $V_\sigma(u) < \infty$ *and* $\|u\|_\infty < \infty$,

† See (**8**). A few intermediary steps can be omitted by consulting (**12**) 43.

since § 7.4 then yields $\Gamma(v) \in \mathscr{A}_p$ and $c = \Gamma(w) \in \mathscr{A}_p$; from § 7.3 we see that $j \circ c = j \circ \Gamma(w)$ also belongs to $\mathscr{A}_p$, and (8) now shows that $a \in \mathscr{A}_p$; having thus found that $\mathscr{R}_\sigma \subseteq \mathscr{A}_p$, we conclude the proof by referring to § 6.3. To establish (f), note first that, in either case,

$$u_\nu = \lambda(a_\nu + \mu a_{-\nu})$$

with $|\lambda| = |\mu| = 1$, so that

$$\|u\|_\infty \leqslant 2\|a\|_\infty < \infty$$

and $$|[\Delta u]_\nu| = |\lambda([\Delta a]_\nu - \mu[\Delta a]_{-\nu-1})| \leqslant |[\Delta a]_\nu| + |[\Delta a]_{-\nu-1}| = \alpha_\nu;$$

from (7) the conclusion now follows that

$$V_\sigma(u) \leqslant V_\sigma(a) < \infty.$$

8. The main results

The set of all sequences of bounded variation [(**5**) 230] is simply $\mathscr{R}_u$ when $u = (u_\iota)$ $(\iota \in \Omega)$, $u_\iota = \{\nu \in \Omega\colon |\nu| \leqslant \iota\}$.† Clearly, $\mathscr{R}_u \subseteq \mathscr{R}_\sigma$, and § 7.5 now implies that $\mathscr{R}_u \subseteq \mathscr{A}_p$; a reference to § 6.2 and § 6.3 shows that the properties (i), (ii), (iii) are satisfied in the case $1 < p < \infty$ and $a \in \mathscr{R}$, when $\mathscr{R} = \mathscr{R}_u$ or $\mathscr{R} = \mathscr{R}_\sigma$.‡

Suppose that $$A \in L^1 = L^1[-\pi, \pi], \qquad x \in L^1,$$

and let f and g be periodic functions equivalent to A and x respectively. We can then define $A * x$ as the function z such that

$$z(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta - \lambda) g(\lambda)\, d\lambda \quad (-\pi \leqslant \theta \leqslant \pi).$$

It is well known [(**2**) 23] that

$$(\Phi A) \circ (\Phi x) = \Phi y, \qquad A * x = y \in L^1. \tag{9}$$

Consequently $$(\Phi A) \circ (\Phi x) \in \Phi(L^1)$$

for all x in L^1, and $\Phi A \in \mathscr{A}_1$. In view of § 4.3, this implies that

$$a = \Phi A \in \mathscr{A}_p,$$

and $$\Phi(L^1) \subseteq \mathscr{A}_1 = \mathscr{A}_\infty \subseteq \mathscr{A}_p = \mathscr{A}_{p'} \subseteq \mathscr{B}.$$

If $x \in L^p$, it can be inferred from (9) that

$$a \circ (\Phi x) = \Phi(A * x),$$

† See also § 6.1 and footnote ‡ of p. 1.

‡ Some of the results which concern the Banach algebra $\mathscr{R}_u$ are mentioned in (**10**) 491 and (**5**) 230; it may be of interest to note that $\mathscr{R}_u$ is a separable subspace of the Banach algebra $\mathscr{R}_\sigma$.

and from § 3.3 (d) that

$$\Phi(A * x) = \Phi(T_{a,p}x),$$

so that $A * x = T_{a,p}x$. Accordingly, if $a = \Phi A$, then

$$A_{*p} = \{x \in L^p \to A * x\} = T_{a,p},$$

and A_{*p} is a bounded linear operator with point-spectrum $\{a_n : n \in \Omega\}$.

If A is restricted to the set (AC) of all absolutely continuous functions periodic on $[-\pi, \pi]$, then $[\Phi A]_n = o(|n|^{-1})$ [see (**2**) 23], so that

$$(\Phi A) \in \mathscr{R}_\sigma,$$

in view of § 7.1. In the preceding paragraph we saw that $A_{*p} = T_{a,p}$ when $a = \Phi A$; it can now be inferred from § 7.5 and $(\Phi A) \in \mathscr{R}_\sigma$ that

If $A \in (AC)$, the point-spectrum of A_{*p} is the range $\{[\Phi A]_n : n \in \Omega\}$ of ΦA, and the spectrum of A_{*p} is the closure of its point-spectrum; moreover, for any p with $1 < p < \infty$,

$$\|\Phi A\|_\infty = \lim_{\nu \to \infty} \|(A_{*p})^\nu\|_p^{1/\nu} \leqslant \|A_{*p}\|_p \leqslant \|A\|_1.$$

The last inequality is an easy consequence of Hölder's. There is no need to restrict A to (AC) in order to ensure the above properties for $p = 2$, since we then have $\Phi A \in \mathscr{A}_2$ [see § 6].

8.1. *Remark.* It is easily verified that the set $\mathscr{R} = \mathscr{A}_1 = \mathscr{A}_\infty$ forms a Banach algebra under the norm $\|a\| = \|T_{a,1}\|_1$. Fom § 5.1 it therefore follows that $(f(a_n))$ $(n \in \Omega)$ is in $\mathscr{R}$ when $a \in R$ and provided that f is analytic on $\sigma(a, \mathscr{R})$; this is a slight improvement of a result mentioned by Hille [(**4**) 17.2.3], which requires that f be analytic on the set $\{\lambda : |\lambda| \leqslant \|a\|\}_\smile \supseteq \sigma(a, \mathscr{R})$ [see (**11**) 406].

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ON Q-KERNELS WITH OPERATORS

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1. Introduction

LET the multiplicative group P operate (on the left) as a group of automorphisms of the additive abelian group G.

A (left) *distributive space* over (P, G) is a set D, together with a left multiplicative operation of P on D and a left additive simply transitive operation of G on D, such that

$$p(g+d) = pg+pd \tag{1.1}$$

for all $p \in P$, $g \in G$, $d \in D$.

In § 2 we show that the set of all isomorphism-classes of non-empty distributive spaces, over a fixed pair (P, G), can be made into a group which is isomorphic in a natural way to the cohomology group $H^1(P, G)$. The 'zero' distributive space is exemplified by taking $D = G$, with G acting on itself by left translation, and P acting on G as given. An arbitrary distributive space over (P, G) is isomorphic to the 'zero' one if and only if it contains a point which is fixed under all elements of P.

Now let Q and K be arbitrary groups, which we shall write additively throughout Part I, although neither is assumed to be abelian. Let $A(K)$ and $I(K)$ denote, respectively, the group of all automorphisms of K and the group of all inner automorphisms of K; let $O(K) = A(K)/I(K)$. Let $\theta: Q \to O(K)$ be a Q-kernel (**5**). Two extensions E and E' of the Q-kernel are called *isomorphic* (or *equivalent*) if there exists an isomorphism of the group E onto the group E' which induces the identity maps of K and Q on themselves. It is sometimes necessary to consider the broader equivalence relation which results when the isomorphism of E onto E' is allowed to induce non-trivial automorphisms of Q and/or K.

Suppose P to be a multiplicative group which acts as a group of automorphisms on both Q and K, in a manner compatible with $\theta: Q \to O(K)$. There results an operation (§ 3) of P on the set of all extensions of the Q-kernel, and indeed on the set EXT of all isomorphism-classes of extensions. The broader classification of extensions suggested above can be subsumed under the problem: *classify the orbits in EXT under the operation of an arbitrary group P.*

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The operations of P on Q and K induce a natural operation of P on the cohomology groups $H^n(Q, Z_K)$, where Z_K denotes the centre of K. It was incorrectly asserted in (**2**) that (at least when the operation of P on K is trivial) the orbits in EXT under P can be put into one-to-one correspondence with the orbits in $H^2(Q, Z_K)$ under P. A counter-example is given in Part III of the present paper.

The proper formulation of the classification (**5**) of EXT in terms of $H^2(Q, Z_K)$ is that the latter operates in a natural way as a simply transitive permutation group on the former. In § 3 we show that EXT is actually a distributive space over the pair $[P, H^2(Q, Z_K)]$. By § 2, the structure of this distributive space is completely determined by the corresponding element of $H^1[P, H^2(Q, Z_K)]$. This element is called the *deviation* of (θ, P); it is the 'obstruction' to the existence of an extension of the Q-kernel which is transformed to an isomorphic extension by every element of P. If we take P to be the group of all

$$(p_Q, p_K) \in A(Q) \times A(K)$$

which satisfy (3.10), the deviation may be regarded as a new invariant of the Q-kernel $\theta: Q \to O(K)$; it is defined only when the Eilenberg–MacLane invariant (**5**) vanishes, so that the distributive space EXT is non-empty.

In Part II we indicate relationships between the deviation of (θ, P) and a number of other invariants, old and new, some of which are defined whether EXT is empty or not. The main result of Part II (Theorem 5) asserts that an element of a group of the form $H^1[P, H^2(Q, G)]$ may be 'realized' as the deviation of an extendible (θ, P), with $G = Z_K$ and P acting trivially on K, if and only if the element lies in the image of a certain natural homomorphism

$$\gamma: H_0^3(\bar{Q}, G) \to H^1[P, H^2(Q, G)]. \tag{1.2}$$

Here, $H_0^3(\bar{Q}, G)$ is the kernel of the homomorphism $H^3(\bar{Q}, G) \to H^3(Q, G)$ induced by the natural homomorphism of Q onto $\bar{Q} = Q/Q_0$; Q_0 is the subgroup of Q generated by all elements of the form $pq-q$. The example mentioned above, in which none of the new invariants vanishes, is obtained in Part III by explicitly computing the homomorphism (1.2) for all cyclic groups P and Q.

In general we shall follow the notations of §§ 4 and 5 of (**6**). In particular, $\nu: A(K) \to O(K)$ and $c_K: K \to I(K)$ will denote the natural homomorphisms, the latter being defined by $(c_K k)k' = k+k'-k$; mappings will be multiplied by composition, added by addition or multiplication of values; cochains with non-abelian coefficients are

allowed, but are always assumed normalized. If $f \in C^m[P, C^n(Q, K)]$, then, as in (8), we shall sometimes write $f(p_1,\dots,p_m,q_1,\dots,q_n)$ for

$$[f(p_1,\dots,p_m)](q_1,\dots,q_n).$$

Part I

2. Classification of distributive spaces

Let P and G be as at the beginning of § 1. If G operates simply transitively on D, and d and d' are elements of D, then $d-d'$ will denote that element g of G such that $g+d' = d$.

Given a distributive space D over (P, G), define $\lambda: D \to C^1(P, G)$ by

$$(\lambda d)(p) = pd-d. \tag{2.1}$$

One verifies easily that $\lambda(D) \subset Z^1(P, G)$, and that

$$\lambda(g+d) = \delta g+\lambda d \tag{2.2}$$

for all $g \in G$ and $d \in D$, where, on the right-hand side, g is regarded as an element of $C^0(P, G)$. It follows that, if D is non-empty, $\lambda(D)$ consists of exactly one entire cohomology class of 1-cocycles. This element of $H^1(P, G)$ is called the *deviation* of D.

[2.3] *The deviation of a non-empty distributive space D vanishes if and only if D contains a point which is left fixed by every element of P.*

Proof. Since $\lambda(D)$ is an entire cohomology class, the deviation vanishes if and only if there exists $d \in D$ such that $\lambda d = 0$.

[2.4] *Let D be a set on which G acts additively and simply transitively, and let $\lambda: D \to Z^1(P, G)$ be any map satisfying* (2.2). *Then there exists one and only one operation of P on D, as a permutation group, satisfying* (1.1) *and* (2.1).

Proof. Take (2.1), in the form $pd = (\lambda d)(p)+d$, as the definition of pd. Then (1.1) follows from (2.2). Hence, using $\delta\lambda D = 0$, we have

$$\begin{aligned} p(p'd) &= p[(\lambda d)p'+d] = p[(\lambda d)p']+pd = p[(\lambda d)p']+(\lambda d)(p)+d \\ &= (\lambda d)(pp')+d = (pp')d. \end{aligned}$$

Finally, $1.d = d$ because λd is a normalized cochain.

Two non-empty distributive spaces D and D' over (P, G) will be called *isomorphic* if there exists a map $W: D \to D'$ such that

$$W(g+d) = g+Wd, \qquad W(pd) = p(Wd) \tag{2.5}$$

for all $g \in G$, $d \in D$, $p \in P$. Because of the simple transitivity of G on both D and D', such a map is automatically one-to-one onto; its inverse is another isomorphism.

THEOREM 1. *The function which assigns to each non-empty distributive space its deviation induces a one-to-one correspondence between the set of all isomorphism-classes of non-empty left distributive spaces and the group* $H^1(P, G)$.

Proof. Obviously two isomorphic distributive spaces have the same deviation. Conversely, suppose the distributive spaces D and D' have the same deviation. Then we can choose $d_0 \in D$ and $d'_0 \in D'$ such that $\lambda d_0 = \lambda' d'_0$: that is, such that

$$pd_0 - d_0 = pd'_0 - d'_0$$

for all $p \in P$. It is now easily verified that the map $W: D \to D'$ defined by $W(d) = (d - d_0) + d'_0$ satisfies (2.5).

Given any $h \in H^1(P, G)$, choose $f \in Z^1(P, G)$ such that $f + B^1(P, G) = h$. Choose a set D and a one-to-one correspondence $T: G \to D$. Let G operate on D by

$$g + d = T(g + T^{-1}d);$$

define $\lambda: D \to Z^1(P, G)$ by $\lambda(d) = \delta T^{-1}d + f$. Then

$$\lambda(g+d) = \delta T^{-1}(g+d) + f = \delta(g + T^{-1}d) + f = \delta g + \delta T^{-1}d + f = \delta g + \lambda d,$$

verifying (2.2). Therefore, by [2.4], there exists an operation of P on D such that D becomes a distributive space whose deviation is

$$\lambda(D) = \lambda[T(0)] + B^1(P, G) = f + B^1(P, G) = h.$$

Because of the naturality of Theorem 1, the group structure of $H^1(P, G)$ can be transplanted to the set of all isomorphism-classes of non-empty distributive spaces, which is then called, by abuse of language, 'the group of distributive spaces'. If the distributive spaces D and D' are identical as sets, and G operates in the same way on both, there still appears to be no way of defining the 'sum' $D + D'$ without choosing an arbitrary base point; distinct base points yield distinct (though isomorphic) distributive spaces as the 'sum'.

It can be seen that the group of all automorphisms [in the sense of (2.5)] of any fixed non-empty distributive space is naturally isomorphic to $H^0(P, G)$.

3. The distributive space of extensions

Let $\theta: Q \to O(K)$ be a Q-kernel. As indicated in § 1, we shall call two extensions

$$0 \to K \xrightarrow{\sigma} E \xrightarrow{\rho} Q \to 0, \tag{3.1}$$

$$0 \to K \xrightarrow{\sigma'} E' \xrightarrow{\rho'} Q \to 0 \tag{3.2}$$

of $\theta\colon Q \to O(K)$ *similar* if there exist a homomorphism $W\colon E \to E'$, and automorphisms α and β of K and Q respectively, such that the diagram

$$\begin{array}{ccccccc} 0 \to & K & \xrightarrow{\sigma} & E & \xrightarrow{\rho} & Q & \to 0 \\ & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle W} & & \downarrow{\scriptstyle \beta} & \\ 0 \to & K & \xrightarrow{\sigma'} & E' & \xrightarrow{\rho'} & Q & \to 0 \end{array} \tag{3.3}$$

is commutative; such a W is automatically an isomorphism of E onto E'.

Given an exact sequence (3.1), $C_\sigma\colon E \to A(K)$ will denote the homomorphism defined by

$$(C_\sigma e)k = \sigma^{-1}(e + \sigma k - e). \tag{3.4}$$

Then we recall that the exact sequence (3.1) is an extension of

$$\theta\colon Q \to O(K)$$

if and only if $\theta\rho = \nu C_\sigma$. If (3.1) and (3.2) are such extensions, and (3.3) is commutative, one verifies easily, for each $e \in E$, that

$$C_{\sigma'} We = \alpha(C_\sigma e)\alpha^{-1}. \tag{3.5}$$

[3.6] *If* (3.3) *is a similarity between two extensions of* $\theta\colon Q \to O(K)$, *then for each* $q \in Q$,

$$\theta\beta(q) = \nu(\alpha) + \theta(q) - \nu(\alpha).$$

Proof. Choose $e \in E$ such that $\rho e = q$. Then

$$\theta\beta(q) = \theta\beta\rho e = \theta\rho' We = \nu C_{\sigma'} We = \nu[\alpha(C_\sigma e)\alpha^{-1}]$$

$$= \nu(\alpha) + \nu C_\sigma(e) - \nu(\alpha) = \nu(\alpha) + \theta\rho e - \nu(\alpha) = \nu(\alpha) + \theta(q) - \nu(\alpha).$$

Now let P be a multiplicative group which acts multiplicatively as a group of automorphisms on K and also on Q. For each $p \in P$, p_K and p_Q will denote the resulting automorphisms of K and Q respectively.

If $p \in P$ and the sequence (3.1) is exact, then so is

$$0 \to K \xrightarrow{\sigma p_K^{-1}} E \xrightarrow{p_Q \rho} Q \to 0. \tag{3.7}$$

Let us denote (3.1) by (σ, E, ρ), and (3.7) by $p(\sigma, E, \rho)$. Thus, by definition,

$$p(\sigma, E, \rho) = (\sigma p_K^{-1}, E, p_Q \rho). \tag{3.8}$$

[3.9] *If* $\theta\colon Q \to O(K)$ *is extendible, then the following are equivalent:* (*a*) *there exists an extension* e *of* $\theta\colon Q \to O(K)$ *such that for every* $p \in P$, pe *is an extension of* $\theta\colon Q \to O(K)$; (*b*) *for every extension* e *of* $\theta\colon Q \to O(K)$ *and every* $p \in P$, pe *is an extension of* $\theta\colon Q \to O(K)$; (*c*) *for every* $p \in P$ *and every* $q \in Q$,

$$\theta(p_Q q) = \nu(p_K) + \theta(q) - \nu(p_K). \tag{3.10}$$

Proof. Suppose (*c*) true; given $p \in P$ and an extension (σ, E, ρ) of $\theta\colon Q \to O(K)$. Then $(p_K, 1, p_Q)$ is a similarity from (σ, E, ρ) to $p(\sigma, E, \rho)$. Hence, by (3.5),

$$\nu C_{\sigma p_K^{-1}}(e) = \nu[p_K(C_\sigma e)p_K^{-1}] = \nu(p_K)+\nu C_\sigma e-\nu(p_K)$$
$$= \nu(p_K)+\theta\rho e-\nu(p_K) = \theta p_Q \rho(e).$$

Therefore $(\sigma p_K^{-1}, E, p_Q \rho)$ is an extension of $\theta\colon Q \to O(K)$. Thus (*c*) implies (*b*). By [3.6], (*a*) implies (*c*).

We assume henceforth that (3.10) holds for all $p \in P$ and all $q \in Q$. Then, by [3.9], (3.8) defines a left operation of P on the set of all extensions of $\theta\colon Q \to O(K)$.

One verifies easily

[3.11] *If $p \in P$, and the extensions e and e' of $\theta\colon Q \to O(K)$ are isomorphic, then pe is isomorphic to pe'.*

Hence

[3.12] *The formula* (3.8) *induces a left operation of P on the set EXT of all isomorphism-classes of extensions of $\theta\colon Q \to O(K)$.*

$\theta\colon Q \to O(K)$ induces an operation of Q on Z_K, *via* the natural homomorphism $\eta\colon O(K) \to A(Z_K)$. Since we are writing both Q and Z_K additively, we shall also write this operation additively. Thus, for each $q \in Q$ and $z \in Z_K$,

$$q+z = (\eta\theta q)(z). \tag{3.13}$$

From (3.10) it now follows that

$$p_K(q+z) = p_Q q+p_K z \tag{3.14}$$

for each $p \in P$, $q \in Q$, and $z \in Z_K$.

Given $p \in P$ and $f \in C^n(Q, K)$, define $pf \in C^n(Q, K)$ by

$$(pf)(q_1,\dots,q_n) = p_K[f(p_Q^{-1}q_1,\dots,p_Q^{-1}q_n)]. \tag{3.15}$$

Using (3.14), we obtain

$$\delta(pf) = p(\delta f) \tag{3.16}$$

for $p \in P$ and $f \in C^n(Q, Z_K)$. Hence (3.15) induces an operation of P on the cohomology groups $H^n(Q, Z_K)$.

By (**5**), as formulated in (**6**), $H^2(Q, Z_K)$ operates simply transitively on EXT.

THEOREM 2. *EXT forms a distributive space over the pair*

$$[P, H^2(Q, Z_K)].$$

Proof. We shall adhere to the terminology and viewpoint of §§ 4, 5 of (**6**). We define an operation of P on the group $C^n[Q, A(K)]$ by

$$(pf)(q_1,\dots,q_n) = p_K[f(p_Q^{-1}q_1,\dots,p_Q^{-1}q_n)]p_K^{-1}. \tag{3.17}$$

Given $p \in P$, and a structural cochain (f,t) in $\theta\colon Q \to O(K)$, we define

$$p(f,t) = (pf, pt), \tag{3.18}$$

where the right-hand side is defined by (3.15) and (3.17). Then $p(f,t)$ is a structural cochain in $\theta\colon Q \to O(K)$; moreover, for each $h \in C^1(Q,K)$,

$$\delta[p(f,t)] = p[\delta(f,t)], \qquad p(h_t.f) = (ph)_{pt}.(pf), \tag{3.19}$$
$$p(c_K h) = c_K(ph).$$

It follows that (3.18) induces operations of P on the set $\bar{Z}^2(Q,K,\theta)$ of all structural cocycles and the set $\bar{H}^2(Q,K,\theta)$ of all equivalence classes of structural cocycles. If (f,t) is a structural cocycle, and $g \in Z^2(Q,Z_K)$, then referring to formula (5.11) in (**6**), we have

$$\begin{aligned} p[g+(f,t)] = p(g+f,t) &= [p(g+f), pt] = (pg+pf, pt) \\ &= pg+(pf,pt) = pg+p(f,t). \end{aligned}$$

Consequently (1.1) holds for $p \in P$, $g \in H^2(Q,Z_K)$, and $d \in \bar{H}^2(Q,K,\theta)$; i.e. $\bar{H}^2(Q,K,\theta)$ is a distributive space over the pair $[P, H^2(Q,Z_K)]$.

The operation of $H^2(Q,Z_K)$ on EXT is defined by combining the operation on $\bar{H}^2(Q,K,\theta)$ with the natural one-to-one correspondence

$$\bar{H}^2(Q,K,\theta) \longleftrightarrow EXT. \tag{3.20}$$

Therefore, in order to complete the proof of Theorem 2, it remains only to establish

[3.21] *The operation of P on $\bar{H}^2(Q,K,\theta)$ induced by* (3.18) *corresponds,* via (3.20), *to the operation of P on EXT induced by* (3.8). *That is, if the structural cocycle (f,t) represents the extension (σ, E, ρ), then the structural cocycle $p(f,t)$ represents the extension $p(\sigma, E, \rho)$.*

Proof of [3.21]. Choose $d \in C^1(Q,E)$ such that $\rho d = 1$ and

$$(f,t) = (\sigma^{-1}\operatorname{dev} d, C_\sigma d).$$

Let $d' = dp_Q^{-1}$. Then

$$(p_Q\rho)d' = 1,$$

$$\begin{aligned} (\sigma p_K^{-1})^{-1}(\operatorname{dev} d')(q,q') &= p_K\sigma^{-1}[\operatorname{dev}(dp_Q^{-1})(q,q')] \\ &= p_K[(\sigma^{-1}\operatorname{dev} d)(p_Q^{-1}q, p_Q^{-1}q')] \\ &= p_K[f(p_Q^{-1}q, p_Q^{-1}q')] = (pf)(q,q'), \end{aligned}$$

$$\begin{aligned} C_{\sigma p_K^{-1}}d'(q) = (C_{\sigma p_K^{-1}})(dp_Q^{-1}q) &= p_K[C_\sigma(dp_Q^{-1}q)]p_K^{-1} \\ &= p_K[t(p_Q^{-1}q)]p_K^{-1} = (pt)(q). \end{aligned}$$

Therefore (pf, pt) represents $(\sigma p_K^{-1}, E, p_Q\rho)$.

In later sections, the given system, consisting of $\theta\colon Q \to O(K)$ and P operating on K and Q, satisfying (3.10), will sometimes be denoted by (θ, P). The deviation of the distributive space EXT will also be called the deviation of (θ, P), and will be denoted $\chi_1 \in H^1[P, H^2(Q, Z_K)]$.

4. Augmented structural cochains

If (f,t) is a structural cocycle in $\theta\colon Q \to O(K)$, and $p \in P$, we cannot directly 'subtract' (f,t) from $p(f,t)$ in order to obtain the deviation unless $pt = t$. In this section we establish a systematic convention for replacing (f,t) by an equivalent structural cocycle whose second co-ordinate is pt.

An *augmented structural cochain* in (θ, P) is a triple (f,t,w), where (f,t) is a structural cochain in $\theta\colon Q \to O(K)$, $w \in C^1[P, C^1(Q,K)]$, and†

$$c_K[w(p)] = pt - t, \quad \text{for each } p \in P. \tag{4.1}$$

If, in addition, $\delta_t f = 0$, we call (f,t,w) an *augmented structural cocycle*.

It follows from (3.10) that, if (f,t) is a structural cochain and $p \in P$, then $\nu t = \nu(pt)$. Hence every structural cochain can be augmented. (θ, P) admits an augmented structural cocycle if and only if the Q-kernel is extendible.

For each augmented structural cochain (f,t,w), define

$$[d_1(f,t,w)](p) = pf - (wp)_t . f, \quad \text{for each } p \in P. \tag{4.2}$$

Then it is easily verified that

$$d_1(f,t,w) \in C^1[P, C^2(Q, Z_K)],$$

and that

$$\delta([d_1(f,t,w)]p) = p\delta(f,t) - \delta(f,t) \tag{4.3}$$

for each $p \in P$. Hence

$$d_1(f,t,w) \in C^1[P, Z^2(Q, Z_K)]$$

if (f,t,w) is an augmented structural *cocycle*. Let

$$\omega\colon Z^2(Q, Z_K) \to H^2(Q, Z_K)$$

denote the natural homomorphism.

[4.4] *If (f,t,w) is an augmented structural cocycle in (θ, P), then $\omega d_1(f,t,w) \in Z^1[P, H^2(Q, Z_K)]$; and this cocycle represents the deviation of (θ, P).*

We omit the proof.

† In the right-hand side of (4.1), we write $C^1[Q, A(K)]$ additively, and use (3.17). Thus (4.1) means

$$c_K[w(p,q)] = p_K[t(p_Q^{-1} q)]p_K^{-1}(tq)^{-1}.$$

PART II

5. Some cohomology groups and homomorphisms

In this section we assume that Q is an additive group (not necessarily abelian) operating additively as a group of automorphisms on the additive abelian group G and that P is a multiplicative group operating multiplicatively as a group of automorphisms on both Q and G so as to satisfy

$$p_G(q+g) = p_Q q + p_G g \tag{5.1}$$

for all $p \in P$, $q \in Q$, and $g \in G$. In the applications, G will be Z_K [cf. (3.14)].

Let $A^{m,n} = C^m[P, C^n(Q, G)]$; let A denote the direct sum of all the groups $A^{m,n}$ as m and n range through all non-negative integers. Let A^n denote the direct sum of all the groups $A^{i,j}$ for $i+j = n$. For each $a \in A$, $a^{i,j}$ will denote the component of a lying in the group $A^{i,j}$; and a^n will denote the sum of all the components $a^{i,j}$ for $i+j = n$.

We define δ_P: $A^{m,n} \to A^{m+1,n}$ by the standard coboundary formula for 'left operators' (**3**), using the left operation of P on $C^n(Q, G)$ defined by (3.15). The standard coboundary operator δ: $C^n(Q, G) \to C^{n+1}(Q, G)$ induces, by composition with mappings of P^m into $C^n(Q, G)$, a homomorphism of $A^{m,n}$ into $A^{m,n+1}$, which we shall denote by δ_Q. Then $\delta_P \delta_Q = \delta_Q \delta_P$. Hence, except for conventions regarding signs, $\{A, \delta_P, \delta_Q\}$ is a double complex in the sense of (**1**). As in (**8**), we define the 'total coboundary operator' D: $A \to A$ by

$$D(a^{i,j}) = \delta_P a^{i,j} + (-1)^i \delta_Q a^{i,j}. \tag{5.2}$$

Let $Z^n(A) = A^n \cap D^{-1}(0)$, $B^n(A) = D(A^{n-1})$, $H^n(A) = Z^n(A)/B^n(A)$.

The statement $(i,j) \geqslant (m,n)$ will mean that $i \geqslant m$ and $j \geqslant n$; the same convention will apply with pairs replaced by n-tuples. Let $F^{m,n}$ denote the direct sum of all the groups $A^{i,j}$ for $(i,j) \geqslant (m,n)$. In particular, $F^{0,0} = A$ and $F^{\infty,\infty} = 0$. If

$$(0,0) \leqslant (a,b) \leqslant (c,d) \leqslant (\infty,\infty),$$

D: $A \to A$ induces a coboundary operator $F^{a,b}/F^{c,d} \to F^{a,b}/F^{c,d}$, with respect to which the groups $Z^n(F^{a,b}/F^{c,d})$, $B^n(F^{a,b}/F^{c,d})$, $H^n(F^{a,b}/F^{c,d})$ are defined as usual.

Guided by (**8**), we define the 'vector cohomology groups' $\mathbf{H}_k^n(P, Q; G)$ by

$$\mathbf{H}_k^n(P, Q; G) = H^n(F^{0,n-k}). \tag{5.3}$$

The definition of the coboundary operators δ_P and δ_Q in (**8**) appears different. But define ϕ: $A^{m,n} \to A^{m,n}$ by

$$(\phi f)(p_1, \ldots, p_m, q_1, \ldots, q_n) = f(p_1, \ldots, p_m, q_1', \ldots, q_n'), \tag{5.4}$$

where $q_i' = (p_1 \cdots p_m)_Q(q_i)$. Then $\phi: A \to A$ is an isomorphism of the double complex defined above onto the double complex used in (**8**). Under this isomorphism, our definitions of the 'vector cohomology groups' and the 'Whitehead obstruction' (§ 6) correspond to those given in (**8**).

An octet $(a,b,\ldots,h)$ of non-negative integers or ∞ will be called *admissible* if

$$(c,d) \geqslant (a,b) \geqslant (e,f), \qquad (c,d) \geqslant (g,h) \geqslant (e,f).$$

In this case, the inclusions $F^{a,b} \subset F^{e,f}$ and $F^{c,d} \subset F^{g,h}$ induce a homomorphism

$$H^n(F^{a,b}/F^{c,d}) \to H^n(F^{e,f}/F^{g,h}), \tag{5.5}$$

whose image we shall denote by

$$L^n\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}. \tag{5.6}$$

In particular,

$$\mathbf{H}^n_k(P,Q;G) = L^n\begin{pmatrix} 0 & n-k & \infty & \infty \\ 0 & n-k & \infty & \infty \end{pmatrix}. \tag{5.7}$$

Suppose that $(a,b,\ldots,h)$ and $(a',b',\ldots,h')$ are admissible octets, and that $(a,b,\ldots,h) \geqslant (a',b',\ldots,h')$. Then the inclusions $F^{e,f} \subset F^{e',f'}$ and $F^{g,h} \subset F^{g',h'}$ induce a homomorphism

$$L^n\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \to L^n\begin{pmatrix} a' & b' & c' & d' \\ e' & f' & g' & h' \end{pmatrix}, \tag{5.8}$$

whose image is $L^n\begin{pmatrix} a & b & c & d \\ e' & f' & g' & h' \end{pmatrix}$.

[5.9] *Let $(a,b,\ldots,h)$ and $(a',b',\ldots,h')$ be admissible octets, with*

$$(a,b,\ldots,h) \geqslant (a',b',\ldots,h').$$

Suppose that $e = e'$, $(c,d) \geqslant (v,w) \geqslant (g,h)$,

and $\min(b,w,n-g') \geqslant u \geqslant f \geqslant h'$. *Then the kernel of* (5.8) *is*

$$L^n\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \cap L^n\begin{pmatrix} g' & u & v & w \\ e & f & g & h \end{pmatrix}.$$

(We omit the proof.)

Our next immediate objective is the set of formulae (5.12)–(5.18). These express various familiar cohomology groups in terms of the notation $L^n\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$, so that homomorphisms connecting them may be defined as appropriate instances of (5.8).

We note first that the two spectral sequences associated with the

double complex A in the manner of [(**1**) 330–1] can be obtained by defining

$$I_r^{p,q} = L^{p+q}\begin{pmatrix} p & 0 & p+r & 0 \\ p-r+1 & 0 & p+1 & 0 \end{pmatrix}, \tag{5.10}$$

$$II_r^{q,p} = L^{p+q}\begin{pmatrix} 0 & p & 0 & p+r \\ 0 & p-r+1 & 0 & p+1 \end{pmatrix}. \tag{5.11}$$

On setting $(p, q, r) = (i, n-i, 2)$ in (5.10), the formula

$$I_2(A) = H_I H_{II}(A)$$

[(**1**) 331] becomes

$$L^n\begin{pmatrix} i & 0 & i+2 & 0 \\ i-1 & 0 & i+1 & 0 \end{pmatrix} \approx H^i[P, H^{n-i}(Q, G)] \quad (n \geqslant i). \tag{5.12}$$

The inclusion $F^{i,n-i} \subset F^{i-1,n-i}$ induces an isomorphism

$$H^i[P, Z^{n-i}(Q, G)] \approx L^n\begin{pmatrix} i & n-i & \infty & \infty \\ i-1 & n-i & \infty & \infty \end{pmatrix}. \tag{5.13}$$

The inclusion of the direct sum of all the groups $A^{0,n}$ in A induces an isomorphism between the two (single) cochain complexes $\{C(Q, G); \delta_Q\}$ and $\{F^{0,0}/F^{1,0}; D\}$. This yields isomorphisms

$$H^n(Q, G) \approx L^n\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{5.14}$$

For the remainder of this section, the symbols $C^n(Q, G)$, $Z^n(Q, G)$, etc. will generally be abbreviated to C^n, Z^n, etc. Let C_e^n denote the group of all cochains $f \in C^n$ such that for each $p \in P$, $pf = f$. Such cochains will be called *equivariant*. [In a different notation, the cochain pf defined by (3.15) might be written $p_K f p_Q^{-1}$; then the formula $pf = f$ would be written $p_K f = f p_Q$.] Define $Z_e^n = Z^n \cap C_e^n$, $B_e^n = \delta_Q C_e^{n-1}$, $H_e^n = Z_e^n/B_e^n$.

Let $\tilde{C}_e^n$ denote the group of all $f \in C^n$ such that for each $p \in P$,

$$pf - f \in B^n.$$

Define $$\tilde{B}_e^n = \delta_Q \tilde{C}_e^{n-1}, \qquad \tilde{H}_e^n = Z_e^n/\tilde{B}_e^n.$$

Let ${}_0H_e^n$ and ${}_0\tilde{H}_e^n$ denote the respective kernels of the natural homomorphisms $H_e^n \to H^n$ and $\tilde{H}_e^n \to H^n$. Then

$${}_0H_e^n = (B^n \cap C_e^n)/B_e^n, \qquad {}_0\tilde{H}_e^n = (B^n \cap C_e^n)/\tilde{B}_e^n.$$

We note also that

$$H^0(P, C^n) = C_e^n, \qquad H^0(P, B^n) = B^n \cap C_e^n, \qquad H^0(P, Z^n) = Z_e^n.$$

The natural homomorphisms $H^n(F^{0,n}) \to H^n(F^{0,n-1})$ and

$$H^n(F^{0,n}) \to H^n(F^{0,0}/F^{2,0})$$

induce, respectively:

$$H^n_e \approx L^n\begin{pmatrix} 0 & n & \infty & \infty \\ 0 & n-1 & \infty & \infty \end{pmatrix}, \tag{5.15}$$

$$\tilde{H}^n_e \approx L^n\begin{pmatrix} 0 & n & \infty & \infty \\ 0 & 0 & 2 & 0 \end{pmatrix}. \tag{5.16}$$

Using [5.9], we obtain

$${}_0H^n_e \approx L^n\begin{pmatrix} 0 & n & \infty & \infty \\ 0 & n-1 & \infty & \infty \end{pmatrix} \cap L^n\begin{pmatrix} 1 & n-1 & \infty & \infty \\ 0 & n-1 & \infty & \infty \end{pmatrix}, \tag{5.17}$$

$${}_0\tilde{H}^n_e \approx L^n\begin{pmatrix} 0 & n & \infty & \infty \\ 0 & 0 & 2 & 0 \end{pmatrix} \cap L^n\begin{pmatrix} 1 & n-1 & \infty & \infty \\ 0 & 0 & 2 & 0 \end{pmatrix}. \tag{5.18}$$

Setting $i = 1$ in (5.13) and comparing with (5.17), we obtain a homomorphism

$${}_0H^n_e \to H^1(P, Z^{n-1}), \tag{5.19}$$

with kernel zero. Similarly, comparing (5.12) with (5.18), and noting that $L^n\begin{pmatrix} 1 & n-1 & \infty & \infty \\ 0 & 0 & 2 & 0 \end{pmatrix}$ is a subgroup of $L^n\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$, we obtain a homomorphism

$${}_0\tilde{H}^n_e \to H^1(P, H^{n-1}), \tag{5.20}$$

with kernel zero.

On the other hand, the exact sequence

$$0 \to Z^r \to C^r \xrightarrow{\delta_Q} B^{r+1} \to 0 \tag{5.21}$$

of operator homomorphisms (relative to P) induces, in the usual manner, an exact sequence

$$\ldots \to H^k(P, Z^r) \to H^k(P, C^r) \to H^k(P, B^{r+1}) \xrightarrow{\nabla} H^{k+1}(P, Z^r) \to \ldots. \tag{5.22}$$

Hence ∇ induces a homomorphism

$$H^k(P, B^{r+1})/\delta_Q H^k(P, C^r) \to H^{k+1}(P, Z^r), \tag{5.23}$$

with kernel zero. Setting $k = 0$ and $r = n-1$, we obtain

$$\psi\colon {}_0H^n_e \to H^1(P, Z^{n-1}). \tag{5.24}$$

It can be verified that the kernel of the composition

$$H^0(P, B^n) \xrightarrow{\nabla} H^1(P, Z^{n-1}) \xrightarrow{\omega} H^1(P, H^{n-1}) \tag{5.25}$$

is $\tilde{B}^n_e$; hence (5.25) induces

$$\tilde{\psi}\colon {}_0\tilde{H}^n_e \to H^1(P, H^{n-1}). \tag{5.26}$$

It can be verified that the sum of the homomorphisms (5.19), (5.24) is zero; and the sum of the homomorphisms (5.20), (5.26) is zero.

For the rest of this section we assume that the operation of P on G

is trivial. As in § 1, let Q_0 denote the subgroup (automatically normal) of Q generated by all elements of the form $p_Q q - q$; let $\bar{Q} = Q/Q_0$, and let $\beta: Q \to \bar{Q}$ denote the natural homomorphism. Formula (5.1) now reads $p_Q q + g = q + g$; hence the operation of Q_0 on G is trivial; hence the operation of Q on G induces an operation of $\bar{Q}$ on G. If we let P operate trivially on $\bar{Q}$, $\beta: Q \to \bar{Q}$ becomes an operator homomorphism relative to both P and G. $\beta: Q \to \bar{Q}$ induces homomorphisms

$$\beta^*: H^n(\bar{Q}, G) \to H^n_e(Q, G), \tag{5.27}$$

which when composed with the natural homomorphisms

$$H^n_e(Q, G) \to H^n(Q, G),$$

yield the usual homomorphisms $H^n(\bar{Q}, G) \to H^n(Q, G)$. The kernel of the latter will be denoted $H^n_0(\bar{Q}, G)$, as in § 1. The composition

$$H^n_0(\bar{Q}, G) \xrightarrow{\beta^*} {}_0H^n_e(Q, G) \xrightarrow{\psi} H^1[P, Z^{n-1}(Q, G)] \xrightarrow{\omega} H^1[P, H^{n-1}(Q, G)] \tag{5.28}$$

will be written as

$$\gamma: H^n_0(\bar{Q}, G) \to H^1[P, H^{n-1}(Q, G)], \tag{5.29}$$

and will be studied in Part III.

6. The Whitehead obstruction

Let $\theta: Q \to O(K)$ and P be as in §§ 3, 4. We shall use the notations of § 5, with $G = Z_K$.

For each augmented structural cochain (f, t, w) (§ 4), define

$$D(f, t, w) \in A^3$$

by $$D(f, t, w) = d_0(f, t, w) + d_1(f, t, w) + d_2(f, t, w),$$

abbreviated to $d_0 + d_1 + d_2$, where $d_0 = \delta_1 f \in A^{0,3}$, $d_1 \in A^{1,2}$ is defined by (4.2), and $d_2 \in A^{2,1}$ is defined by

$$d_2(p, p') = p(wp') + w(p) - w(pp'). \tag{6.1}$$

Let S denote the set of all augmented structural cochains in (θ, P).

[6.2] *$D(S)$ comprises all of a single coset of $B^3(F^{0,1})$ in $Z^3(F^{0,1})$. The isomorphism ϕ defined by* (5.4) *carries $D(S)$ to the invariant defined in* (8) [*pp.* 225–6].

We omit the proof.

$D(S)$, regarded as an element of $H^3(F^{0,1}) = \mathbf{H}^3_2(P, Q; Z_K)$, will be called the *Whitehead obstruction* of (θ, P) and written χ_W. The Eilenberg–MacLane obstruction (**5**) of the Q-kernel $\theta: Q \to O(K)$ will be denoted by $\chi_{EM} \in H^3(Q, Z_K)$. Combining (5.7) and (5.14) with the appropriate instance of (5.8), we obtain a homomorphism $\mathbf{H}^3_2(P, Q; Z_K) \to H^3(Q, Z_K)$,

which (in view of the above definition of d_0) clearly carries χ_W to χ_{EM}. But, by (5.9), the kernel of this homomorphism is the subgroup

$$L^3\begin{pmatrix}1 & 1 & \infty & \infty\\ 0 & 1 & \infty & \infty\end{pmatrix} \subset \mathbf{H}^3_2(P, Q; Z_K). \tag{6.3}$$

Hence the Q-kernel is extendible if and only if $\chi_W \in L^3\begin{pmatrix}1 & 1 & \infty & \infty\\ 0 & 1 & \infty & \infty\end{pmatrix}$.

Combining (5.12) with the appropriate instance of (5.8), we obtain a homomorphism

$$L^3\begin{pmatrix}1 & 1 & \infty & \infty\\ 0 & 1 & \infty & \infty\end{pmatrix} \to H^1[P, H^2(Q, Z_K)]. \tag{6.4}$$

Using [4.4], we obtain

[6.5] *If the Q-kernel $\theta\colon Q \to O(K)$ is extendible, then the homomorphism* (6.4) *carries χ_W to χ_1.*

Return now to P, Q, G as in § 5; let D be a non-empty distributive space over $[P, H^2(Q, G)]$. We say that D is *the distributive space of a Q-kernel* if there exist a group K, an operation of P on K, and a homomorphism $\theta\colon Q \to O(K)$, satisfying (3.10), and such that

$$Z_K = G, \tag{6.6}$$

the induced operation of P on Z_K coincides with the given operation of P on G, (6.7)

the induced operation of Q on Z_K coincides with the given operation of Q on G, (6.8)

the distributive space of extensions of $\theta\colon Q \to O(K)$ is isomorphic to D. (6.9)

THEOREM 3. *A non-empty distributive space D over $[P, H^2(Q, G)]$ is the distributive space of a Q-kernel if and only if its deviation lies in the image of the homomorphism* (6.4), *with Z_K replaced by G.*

Proof. The 'only if' part follows from [6.5]. Conversely, suppose that $h \in H^1[P, H^2(Q, G)]$ is the image of

$$g \in L^3\begin{pmatrix}1 & 1 & \infty & \infty\\ 0 & 1 & \infty & \infty\end{pmatrix} \subset \mathbf{H}^3_2(P, Q; G).$$

By the penultimate paragraph of (**8**), there exist a group K, an operation of P on K, and a homomorphism $\theta\colon Q \to O(K)$ satisfying (3.10) and (6.6)–(6.8), and such that $\chi_W = g$. Hence the Q-kernel is extendible; by [6.5], $\chi_1(\theta, P) = h$; whence (6.9) follows by Theorem 1 (§ 2).

7. The invariants χ'_1, χ'_2, and $\bar{\chi}$

Let $\theta\colon Q \to O(K)$ and P be as in §§ 3, 4, 6. *For the rest of the paper, we assume that the operation of P on K is trivial*: $p_K k = k$ for all $p \in P$ and $k \in K$. In the present section, however, we do not assume that the Q-kernel $\theta\colon Q \to O(K)$ is extendible.

Given any group X on which P operates (trivially or otherwise), the analogue of (3.15) provides an operation of P on $C^n(Q, X)$. $C^n_e(Q, X)$ will denote the group of all $f \in C^n(Q, X)$ such that $pf = f$ for all $p \in P$.

[7.1] *If P operates trivially on the groups X and Y, and the homomorphism $\alpha\colon X \to Y$ is onto, then so is the induced homomorphism*

$$\alpha\colon C^n_e(Q, X) \to C^n_e(Q, Y).$$

Proof. Choose $d \in C^1(Y, X)$ such that $\alpha d = 1$. Then d induces a function $d\colon C^n(Q, Y) \to C^n(Q, X)$, still satisfying $\alpha d = 1$; and, from the triviality of the operation of P on X and Y, we obtain

$$d[C^n_e(Q, Y)] \subset C^n_e(Q, X).$$

Let T denote the set of all cochains $t \in C^1_e[Q, A(K)]$ such that $\nu t = \theta$. Applying [7.1] to $\nu\colon A(K) \to O(K)$ and $c_K\colon K \to I(K)$, we obtain

[7.2] *T is always non-empty. For each $t \in T$, there exists $f \in C^2_e(Q, K)$ such that $c_K f = \operatorname{dev} t$. If $t \in T$ and $t' \in T$, then there exists $h \in C^1_e(Q, K)$ such that $t' = c_K h + t$.*

By Lemmas 5.15 and 5.16 of (**6**) we have

[7.3] *If $\theta\colon Q \to O(K)$ is extendible, then for every $t \in T$ there exists $f \in C^2(Q, K)$, not necessarily in $C^2_e(Q, K)$, such that (f, t) is a structural cocycle.*

A structural cochain (f, t) will be called *invariant* if $f \in C^2_e(Q, K)$ and $t \in C^1_e[Q, A(K)]$. We note that (7.2) enables us to rewrite the proof of Theorem 7.1 of (**5**) [cf. Theorem 5.17 of (**6**)], using invariant cochains throughout; whence

THEOREM 4. *The set of all coboundaries of all invariant structural cochains in $\theta\colon Q \to O(K)$ constitutes a coset of $B^3_e(Q, Z_K)$ in $Z^3_e(Q, Z_K)$.*

The resulting element of $H^3_e(Q, Z_K)$ will be denoted χ'_2. Its image in $\bar{H}^3_e(Q, Z_K)$ will be denoted χ'_1.

We note, by (5.7), that $L^3\begin{pmatrix} 0 & 3 & \infty & \infty \\ 0 & 1 & \infty & \infty \end{pmatrix}$ is a subgroup of $\mathbf{H}^3_2(P, Q; Z_K)$. Combining (5.15) and (5.16) with appropriate versions of (5.8), we obtain homomorphisms

$$H^3_e(Q, Z_K) \to L^3\begin{pmatrix} 0 & 3 & \infty & \infty \\ 0 & 1 & \infty & \infty \end{pmatrix} \to \bar{H}^3_e(Q, Z_K). \tag{7.4}$$

Any invariant structural cochain (f, t) can be made into an augmented structural cochain (§ 4) by setting $w = 0 \in C^1[P, C^1(Q, K)]$. It follows that the first part of (7.4) carries χ_2' to χ_W; hence the second part of (7.4) carries χ_W to χ_1'. In particular, we note that as a consequence of our assumption that the operation of P on K is trivial, χ_W always lies in the subgroup $L^3\begin{pmatrix} 0 & 3 & \infty & \infty \\ 0 & 1 & \infty & \infty \end{pmatrix}$.

It follows from Lemma 5.19 of (**6**) that the vanishing of χ_2' (i.e. the existence of an invariant structural cocycle) is equivalent to the existence of an extension (3.1) of $\theta\colon Q \to O(K)$, together with a cochain $d \in C^1(Q, E)$ such that $\rho d = 1$, and such that for each $p \in P$, the mapping $W_p\colon E \to E$ defined by

$$W_p(\sigma k + dq) = \sigma k + d(pq) \tag{7.5}$$

is an endomorphism of E. In this case, the function $p \to W_p$ is a homomorphism of P into $A(E)$, and (σ, E, ρ, W) is a P-extension in the sense of (**8**). Not all P-extensions, however, are of the kind just described.

Recall the last part of § 5, with $G = Z_K$. Formula (3.10) now reads $\theta(pq) = \theta(q)$; hence $\theta(Q_0)$ is the trivial element of $O(K)$. Therefore $\theta\colon Q \to O(K)$ induces a $\bar{Q}$-kernel $\bar{\theta}\colon \bar{Q} \to O(K)$, whose Eilenberg–MacLane obstruction (**5**) will be denoted by

$$\bar{\chi} \in H^3(\bar{Q}, Z_K). \tag{7.6}$$

Clearly the homomorphism $\beta^*\colon H^3(\bar{Q}, Z_K) \to H^3_e(Q, Z_K)$ carries $\bar{\chi}$ to χ_2'.

Suppose finally that the Q-kernel $\theta\colon Q \to O(K)$ is extendible. This is equivalent to each of the following assumptions:

$$\bar{\chi} \in H^3_0(\bar{Q}, Z_K), \qquad \chi_2' \in {}_0H^3_e(Q, Z_K),$$

$$\chi_W \in L^3\begin{pmatrix} 0 & 3 & \infty & \infty \\ 0 & 1 & \infty & \infty \end{pmatrix} \cap L^3\begin{pmatrix} 1 & 1 & \infty & \infty \\ 0 & 1 & \infty & \infty \end{pmatrix}, \qquad \chi_1' \in {}_0\tilde{H}^3_e(Q, Z_K).$$

[7.7] *If the operation of P on K is trivial, and the Q-kernel is extendible, then*

$$\tilde{\psi}\colon {}_0\tilde{H}^3_e(Q, Z_K) \to H^1[P, H^2(Q, Z_K)]$$

carries χ_1' to $-\chi_1$, and

$$\gamma\colon H^3_0(\bar{Q}, Z_K) \to H^1[P, H^2(Q, Z_K)]$$

carries $\bar{\chi}$ to $-\chi_1$.

The proof, which we omit, is based on the fact that

$$\bar{\chi} \to \chi_2' \to \chi_W \to \chi_1'$$

via $$H^3(\bar{Q}, Z_K) \to H^3_e(Q, Z_K) \to L^3\begin{pmatrix} 0 & 3 & \infty & \infty \\ 0 & 1 & \infty & \infty \end{pmatrix} \to \tilde{H}^3_e(Q, Z_K),$$

together with Proposition [6.5] and the assertion following (5.26).

Since the kernel of $\bar{\psi}$ is zero, it follows that under the hypotheses of [7.7], the invariant χ_1' is 'equivalent' to the invariant χ_1. We note that, while χ_1 measures the non-invariance of an equivalence class of structural cocycles, χ_1' measures the failure of invariant structural cochains to be structural cocycles.

[7.8] *Under the hypotheses of* [7.7], *if Q is a simple group, then $\bar{\chi} = 0$, and hence $\chi_1 = 0$.*

Proof. If $Q_0 = 0$, $H^3(\bar{Q}, Z_K) \to H^3(Q, Z_K)$ is an isomorphism. If $Q_0 = Q$, then $\bar{Q} = 0$. In either case, $H_0^3(\bar{Q}, Z_K) = 0$.

Recall the last part of § 6.

THEOREM 5. *Let P, Q, G be as in § 5, with P operating trivially on G. Let D be a non-empty distributive space over $[P, H^2(Q, G)]$. Then D is the distributive space of some Q-kernel $\theta\colon Q \to O(K)$, with P operating trivially on K, if and only if the deviation of D lies in the image of*

$$\gamma\colon H_0^3(\bar{Q}, G) \to H^1[P, H^2(Q, G)].$$

Proof. The 'only if' part follows from [7.7]. Conversely, suppose the deviation of D is $\gamma(h)$, where $h \in H_0^3(\bar{Q}, G)$. By Lemma 9.1 of (**5**), there exists a $\bar{Q}$-kernel $\bar{\theta}\colon \bar{Q} \to O(K)$ such that $Z_K = G$, the induced operation of $\bar{Q}$ on Z_K coincides with the given (from Q) operation of $\bar{Q}$ on G, and the Eilenberg–MacLane obstruction of $\bar{\theta}\colon \bar{Q} \to O(K)$ is $-h$. Let

$$\theta = \bar{\theta}\beta\colon Q \to O(K),$$

and let P operate trivially on K. Then

$$\theta(pq) = \bar{\theta}\beta(pq) = \bar{\theta}\beta(q) = \theta(q),$$

so that (3.10) is satisfied. Since

$$\bar{\chi} = -h \in H_0^3(\bar{Q}, G),$$

the distributive space of extensions of $\theta\colon Q \to O(K)$ is non-empty. By [7.7], the deviation of this distributive space is $-\gamma(\bar{\chi}) = \gamma(h)$; hence, by Theorem 1, the distributive space of extensions is isomorphic to D.

8. The invariant χ_2

As in the second part of § 7, we assume that the operation of P on K is trivial, and that the Q-kernel $\theta\colon Q \to O(K)$ is extendible.

Recall the definition of T from § 7. For each $t \in T$, let ${}_tS$ denote the set of all $f \in C^2(Q, K)$ such that $c_K f = \operatorname{dev} t$; let ${}_tS^0$ denote the set of all $f \in {}_tS$ such that $\delta_t f = 0$; let

$${}_tS_e = {}_tS \cap C_e^2(Q, K), \qquad {}_tS_e^0 = {}_tS^0 \cap C_e^2(Q, K) = {}_tS^0 \cap {}_tS_e.$$

By [7.2] and [7.3], for each $t \in T$, the sets ${}_tS_e$ and ${}_tS^0$ are non-empty. The following conditions are equivalent: $\chi_2' = 0$; some ${}_tS_e^0$ is non-empty; every ${}_tS_e^0$ is non-empty.

For each $t \in T$, $Z^2(Q, Z_K)$ operates additively on ${}_tS^0$ by left translation: in fact, by Lemma 4.13 of (**6**), the latter is a coset of the former in the group $C^2(Q, K)$. Thus ${}_tS^0$ forms a distributive space over $[P, Z^2(Q, Z_K)]$. Using [7.2], and §§ 4, 5 of (**6**), it can be shown that, if t and t' are elements of T, the distributive spaces ${}_tS^0$ and ${}_{t'}S^0$ are isomorphic. The common deviation of all the distributive spaces ${}_tS^0$, for all $t \in T$, will be written

$$\chi_2(\theta, P) \in H^1[P, Z^2(Q, Z_K)]. \tag{8.1}$$

The homomorphism

$$\psi\colon {}_0H_e^3(Q, Z_K) \to H^1[P, Z^2(Q, Z_K)]$$

carries χ_2' to $-\chi_2$ (cf. [7.7]), and the natural homomorphism

$$\omega\colon H^1(P, Z^2) \to H^1(P, H^2)$$

carries χ_2 to χ_1. Since the kernel of ψ is zero, we may say that χ_2' is 'equivalent' to χ_2. Combining (5.13) and (5.7) with the appropriate instance of (5.8), we obtain a homomorphism

$$H^1[P, Z^2(Q, Z_K)] \to \mathbf{H}_2^3(P, Q; Z_K),$$

which carries χ_2 to χ_W.

Part III

9. Generalized definition of γ

Let P, Q, G be as in § 5; but assume henceforth that the operation of P on G is trivial. We switch to the multiplicative notation for Q, and also for the operation of Q on G. Formula (5.1) now reads

$$qg = (pq)g. \tag{9.1}$$

J will denote the ring of integers, $J(Q)$ the integral group ring of Q, and $\epsilon\colon J(Q) \to J$ the homomorphism defined by

$$\epsilon \sum_i n_i q_i = \sum_i n_i. \tag{9.2}$$

J forms a Q-module [i.e. a $J(Q)$-module] *via* $\epsilon\colon J(Q) \to J$.

As shown in (**1**), the treatment of cohomology groups in (**3**), which we have followed above, is based on the 'standard complex', which is a particular projective resolution of the Q-module J. A special feature of this resolution is that it may be regarded as a functor of Q, in an obvious manner [reflected in (3.15)]. The definition of $H_e^n(Q, G)$ (and hence of γ) in § 5 relies strongly on this functorial property.

On the other hand, the known calculation of the cohomology groups of cyclic groups utilizes what may be called the 'cyclic complex', which is not functorial. Hence it is not possible to compute γ, for cyclic groups, by imitating literally the definition in § 5 within the framework of the cyclic complex.

One possible procedure is to work with a specific homotopy equivalence between the cyclic and standard complexes, such as the one implicit in § 11 of (**3**). Another method, which is preferable on general grounds and will be followed here, is to reformulate the definition of γ in terms of arbitrary projective resolutions, compensating by appropriate homotopies for the loss of the functorial property. Simultaneously, we shall replace the group ring $J(Q)$ by an arbitrary supplemented algebra since this second generalization costs nothing. As a matter of convenience, however, we shall adhere to the description of the *one-dimensional* cohomology group of P in terms of crossed homomorphisms.

In § 13 we give a specific homotopy which measures the 'non-functoriality' of the cyclic complex.

Let K be any commutative ring with identity; let $\alpha\colon \Lambda \to \Lambda'$ be a homomorphism of one supplemented K-algebra (**1**) into another; let $\epsilon\colon \Lambda \to K$ and $\epsilon'\colon \Lambda' \to K$ denote the augmentation homomorphisms; let A and A' be left modules over Λ and Λ' respectively. A K-module homomorphism $f\colon A \to A'$ is called *α-semilinear* if

$$f(\lambda a) = (\alpha\lambda)(fa) \tag{9.3}$$

for all $\lambda \in \Lambda$, $a \in A$. But let ${}_\alpha A'$ denote the K-module A' made into a Λ-module *via* $\alpha\colon \Lambda \to \Lambda'$. Then an α-semilinear map $A \to A'$ is the same thing as a Λ-homomorphism $A \to {}_\alpha A'$. This remark will be used implicitly in applying to semilinear mappings the known facts about linear ones.

Now let the multiplicative group P act as a group of automorphisms of the supplemented K-algebra Λ; and let $\beta\colon \Lambda \to \overline{\Lambda}$ be a fixed supplemented K-algebra homomorphism such that, for all $p \in P$ and all $\lambda \in \Lambda$,

$$\beta(p\lambda) = \beta(\lambda). \tag{9.4}$$

Let G be any $\overline{\Lambda}$-module, regarded also as a Λ-module *via* $\beta\colon \Lambda \to \overline{\Lambda}$. The left operation of P on Λ induces a right operation of P on $H^n(\Lambda, G)$, which we convert to a left operation in the usual manner: $ph = hp^{-1}$. In terms of these data, we shall define a homomorphism

$$\gamma\colon H_0^n(\overline{\Lambda}, G) \to H^1[P, H^{n-1}(\Lambda, G)], \tag{9.5}$$

where $H_0^n(\overline{\Lambda}, G)$ denotes the kernel of the homomorphism

$$H^n(\overline{\Lambda}, G) \to H^n(\Lambda, G)$$

induced by $\beta\colon \Lambda \to \overline{\Lambda}$.

For each $p \in P$, p_Λ will denote the resulting automorphism of Λ. Let X be any Λ-projective resolution of K, and $\overline{X}$ any $\overline{\Lambda}$-projective resolution of K; the boundary operators will be denoted by d and $\bar{d}$ respectively. As in (**1**), we treat $H^n[\mathrm{Hom}_\Lambda(X, G)]$ as a model for $H^n(\Lambda, G)$.†

Regarding $\overline{X}$ as a Λ-complex *via* $\beta\colon \Lambda \to \overline{\Lambda}$, choose a Λ-map $\beta_\sharp\colon X \to \overline{X}$ over the identity map $K \to K$. For each $p \in P$, choose a p_Λ-semilinear map $p_\sharp\colon X \to X$ over the identity map $K \to K$. The ith components of $\beta_\sharp$ and $p_\sharp$ will be denoted by $\beta_i\colon X_i \to \overline{X}_i$ and $p_i\colon X_i \to X_i$ respectively. Now $\beta_\sharp p_\sharp\colon X \to \overline{X}$ is a (βp_Λ)-semilinear map. But, by (9.4), $\beta p_\Lambda = \beta$. Therefore $\beta_\sharp p_\sharp$ is a Λ-map, as is $\beta_\sharp$. Hence we may choose, for each $p \in P$, a Λ-homotopy $s^p\colon X \to \overline{X}$ such that

$$s^p d + \bar{d} s^p = \beta_\sharp p_\sharp - \beta_\sharp. \tag{9.6}$$

Given any element of $H_0^n(\overline{\Lambda}, G)$, choose a representative

$$\bar{h}_n \in \mathrm{Hom}_{\overline{\Lambda}}(\overline{X}_n, G).$$

Then $\bar{h}_n \bar{d}_{n+1} = 0$, and there exists $h_{n-1} \in \hom_\Lambda(X_{n-1}, G)$ such that

$$\bar{h}_n \beta_n = h_{n-1} d_n. \tag{9.7}$$

Choose any such h_{n-1}. Define $g_1\colon P \to \mathrm{Hom}_\Lambda(X_{n-1}, G)$ by

$$g_1(p) = h_{n-1} p_{n-1} - h_{n-1} - \bar{h}_n s^p_{n-1}. \tag{9.8}$$

Then

$$\begin{aligned}(g_1 p) d_n &= h_{n-1} d_n (p_{n-1} - 1) - \bar{h}_n (\beta_n p_n - \beta_n - \bar{d}_{n+1} s^p_n) \\ &= (h_{n-1} d_n - \bar{h}_n \beta_n)(p_{n-1} - 1) = 0.\end{aligned}$$

Thus g_1 maps P into $Z^{n-1}(X, G)$. Define

$$f_1\colon P \to H_\Lambda^{n-1}(X, G) = H^{n-1}(\Lambda, G)$$

by

$$f_1(p) = g_1(p^{-1}) + B_\Lambda^{n-1}(X, G). \tag{9.9}$$

THEOREM 6. *f_1 is a crossed homomorphism. The element of*

$$H^1[P, H^{n-1}(\Lambda, G)]$$

represented by f_1 depends only on the given element of $H_0^n(\overline{\Lambda}, G)$, and not on the choice of X, $\overline{X}$, $\beta_\sharp$, $p_\sharp$, s^p, $\bar{h}_n$, and h_{n-1}. The resulting map (9.5) *is K-linear.*

We omit the proof, but remark that, if the various assertions are numbered in the order of their statement, they are most easily verified in the order 6, 5, 1, 8, 4, 2, 7, 9, 3.

Now suppose that T is a covariant functor which assigns to each

† For $H^n[\mathrm{Hom}_\Lambda(X, G)]$ we shall also write $H_\Lambda^n(X, G) = Z_\Lambda^n(X, G)/B_\Lambda^n(X, G)$.

supplemented K-algebra Λ a Λ-projective resolution $T(\Lambda)$ of K, and to each homomorphism $\alpha\colon \Lambda \to \Lambda'$ of supplemented K-algebras an α-semi-linear map $T(\alpha)\colon T(\Lambda) \to T(\Lambda')$ over the identity $K \to K$ (for example, the standard complex). In the above definition of γ, we may take

$$X = T(\Lambda), \qquad \overline{X} = T(\overline{\Lambda}), \qquad \beta_\sharp = T(\beta), \qquad p_\sharp = T(p_\Lambda),$$
$$s^p = 0.$$

Formulae (9.8) and (9.9) yield

$$f_1(p) = h_{n-1}T(p^{-1})_{n-1} - h_{n-1} + B_\Lambda^{n-1}(T\Lambda, G)$$
$$= ph_{n-1} - h_{n-1} + B_\Lambda^{n-1}(T\Lambda, G) = (\delta_P h_{n-1})(p) + B_\Lambda^{n-1}(T\Lambda, G).$$

Hence, in the situation of § 5, the homomorphism γ defined there coincides with the one defined here.

10. The cyclic resolution

Henceforth we assume Q to be a finite cyclic group of order n; J will continue to denote the ring of integers. The 'cyclic' Q-projective resolution of J (**1**) is defined relative to an arbitrary generator $\mathbf{q}$ of Q as follows. Define

$$D = \mathbf{q} - 1 \in J(Q), \tag{10.1}$$

$$N = \sum_{i=0}^{n-1} \mathbf{q}^i \in J(Q). \tag{10.2}$$

Let $X_i = J(Q)$ for all $i \geqslant 0$. Since 1 is a basis for the Q-module X_i, a Q-homomorphism of X_i into any Q-module may be prescribed by giving its value at 1. For $i > 0$, we define $d_i\colon X_i \to X_{i-1}$ by

$$d_i(1) = \begin{cases} D & \text{if } i \text{ is odd,} \\ N & \text{if } i \text{ is even.} \end{cases} \tag{10.3}$$

Then the complex X together with the augmentation map $\epsilon\colon X_0 \to J$ defined by (9.2) forms a Q-projective resolution of J.

Because of the commutativity of $J(Q)$, D and N act as Q-endomorphisms of any Q-module G by left multiplication. The natural isomorphisms $\mathrm{Hom}_Q(X_i, G) \approx G$ induce isomorphisms

$$H^i(Q, G) \approx \begin{cases} D^{-1}(0)/N(G) & \text{for } i \text{ even and positive,} \\ N^{-1}(0)/D(G) & \text{for } i \text{ odd.} \end{cases} \tag{10.4}$$

[Cf. (**3**), § 11.]

11. Some identities

Continuing from § 10, we let Q' be another finite cyclic group, of order n', with a chosen generator $\mathbf{q}'$; let $\alpha\colon Q \to Q'$ be a group homomorphism. Define the integers t_α, u_α, e_α, r_α, y_α, z_α by the conditions

$$\alpha\mathbf{q} = (\mathbf{q}')^{t_\alpha} \quad (0 \leqslant t_\alpha < n'), \tag{11.1}$$

$$nt_\alpha = n'u_\alpha, \tag{11.2}$$

$$e_\alpha = \text{g.c.d.}(n', t_\alpha), \tag{11.3}$$

$$r_\alpha e_\alpha = n', \qquad y_\alpha e_\alpha = t_\alpha, \qquad z_\alpha r_\alpha = n. \tag{11.4}$$

To prove that z_α is an integer we observe that e_α is the index of $\alpha(Q)$ in Q'; hence r_α is the order of $\alpha(Q)$; hence z_α is the order of the kernel of α.

Define the elements F'_α, L'_α, M'_α of the group ring $J(Q')$ as follows:

$$F'_\alpha = \sum_{i=0}^{t_\alpha - 1} (\mathbf{q}')^i, \tag{11.5}$$

$$L'_\alpha = \sum_{i=0}^{r_\alpha - 1} (\mathbf{q}')^{ie_\alpha}, \tag{11.6}$$

$$M'_\alpha = \sum_{i=0}^{n'-1} m_i (\mathbf{q}')^i, \tag{11.7}$$

where, for each i, the integer m_i is defined by the condition

$$0 \leqslant i - m_i e_\alpha < e_\alpha. \tag{11.8}$$

THEOREM 7 (parts 1–6). $\alpha(D) = D'F'_\alpha$, $\alpha(N) = z_\alpha L'_\alpha$, $D'M'_\alpha = r_\alpha - L'_\alpha$, $L'_\alpha F'_\alpha = y_\alpha N'$, $F'_\alpha(\alpha N) = u_\alpha N'$, $M'_\alpha(\alpha D) = r_\alpha F'_\alpha - y_\alpha N'$.

(The proof is omitted. Parts 5 and 6 are corollaries of parts 1–4.)

Now let $\bar{Q}$ be a third finite cyclic group, of order $\bar{n}$, with a chosen generator $\bar{\mathbf{q}}$; let $\beta\colon Q' \to \bar{Q}$ be a group homomorphism. The above notations may also be applied to the homomorphisms β and $\beta\alpha$. We note that

$$t_{\beta\alpha} \equiv t_\beta t_\alpha \pmod{\bar{n}}, \tag{11.9}$$

$$t_{\beta\alpha} u_\beta u_\alpha = u_{\beta\alpha} t_\beta t_\alpha. \tag{11.10}$$

Define the integer c by

$$c\bar{n} = t_\beta t_\alpha - t_{\beta\alpha}. \tag{11.11}$$

THEOREM 7 (part 7). $(\beta F'_\alpha)\bar{F}_\beta = c\bar{N} + \bar{F}_{\beta\alpha}$. (The proof is omitted.)

For each non-negative integer i we define the rational number σ_i by

$$n\sigma_i = (u_\beta u_\alpha)^i - (u_{\beta\alpha})^i. \tag{11.12}$$

[11.13] *σ_i is an integer, and $\sigma_{i+1} - u_{\beta\alpha}\sigma_i = c(u_\beta u_\alpha)^i$.*

Proof. Use (11.4) as applied to $\beta\alpha$, and (11.11), to 'eliminate' $\bar{n}$, $t_{\beta\alpha}$, n, $t_\beta t_\alpha$. Then (11.10), and (11.2), as applied to $\beta\alpha$, yield

$$u_\beta u_\alpha = z_{\beta\alpha}(cr_{\beta\alpha} + y_{\beta\alpha}), \tag{11.14}$$

$$u_{\beta\alpha} = y_{\beta\alpha} z_{\beta\alpha}. \tag{11.15}$$

The second part of [11.13] follows readily from (11.14) and (11.15); then the first part of [11.13] follows by induction from the second.

12. Induced homomorphisms

Let $\alpha: Q \to Q'$ be as in § 11. Let G be a Q'-module, regarded also as a Q-module *via* $\alpha: Q \to Q'$. In this section we give explicit formulae for the induced homomorphisms

$$\alpha_i^*: H^i(Q', G) \to H^i(Q, G), \tag{12.1}$$

in terms of the models (10.4).

It follows from parts 1 and 5 of Theorem 7 (§ 11) that the formulae

$$\alpha_{2i}(1) = (u_\alpha)^i \qquad (i \geqslant 0), \tag{12.2}$$

$$\alpha_{2i+1}(1) = (u_\alpha)^i F'_\alpha \quad (i \geqslant 0) \tag{12.3}$$

define a Q-map $\alpha_\sharp: X \to X'$ (whose ith component is $\alpha_i: X_i \to X'_i$) over the identity map $K \to K$.

Each element of the group ring $J(Q')$ acts by left multiplication as a Q'-endomorphism (and hence also as a Q-endomorphism) of G. We note that in general, as endomorphisms of G, $D = \alpha(D) \neq D'$, and $N = \alpha(N) \neq N'$. From parts 1 and 5 of Theorem 7 we obtain:

[12.4] *The endomorphism F'_α of G maps $N'^{-1}(0)$ into $N^{-1}(0)$, and $D'(G)$ onto $D(G)$, and therefore induces a homomorphism*

$$(F'_\alpha)_*: N'^{-1}(0)/D'(G) \to N^{-1}(0)/D(G).$$

The endomorphism $u_\alpha: G \to G$ maps $D'^{-1}(0)$ into $D^{-1}(0)$, and $N'(G)$ into $N(G)$, and therefore induces a homomorphism

$$(u_\alpha)_*: D'^{-1}(0)/N'(G) \to D^{-1}(0)/N(G).$$

By (**1**), the homomorphisms (12.1) are induced by *any* Q-map $X \to X'$ over the identity $J \to J$. In particular, they are induced by the map $\alpha_\sharp: X \to X'$ defined by (12.2) and (12.3). Hence

THEOREM 8. *The diagrams*

$$\begin{array}{ccc} H^{2i+1}(Q', G) & \approx & N'^{-1}(0)/D'(G) \\ \downarrow{\scriptstyle \alpha^*_{2i+1}} & & \downarrow{\scriptstyle (u_\alpha)^i (F'_\alpha)_*} \\ H^{2i+1}(Q, G) & \approx & N^{-1}(0)/D(G) \end{array} \tag{12.5}$$

$$\begin{array}{ccc} H^{2i+2}(Q', G) & \approx & D'^{-1}(0)/N'(G) \\ \downarrow{\scriptstyle \alpha^*_{2i+2}} & & \downarrow{\scriptstyle (u_\alpha)^i (u_\alpha)_*} \\ H^{2i+2}(Q, G) & \approx & D^{-1}(0)/N(G), \end{array} \tag{12.6}$$

in which the horizontal isomorphisms are instances of (10.4), *are commutative for all $i \geqslant 0$.*

The rest of this section is devoted to the special case $Q' = Q$, $\mathbf{q}' = \mathbf{q}$. We assume that G is a given Q-module such that ${}_\alpha G = G$; in other

words, that $(\alpha q)g = qg$ for all $q \in Q$ and $g \in G$ [cf. (9.1)]. We regard $H^i(Q, G)$ as a Q-module *via* the operation of Q on G. In particular, the element F_α of $J(Q)$ acts as an endomorphism of $H^i(Q, G)$, by left multiplication.

THEOREM 9. *For all* $i \geqslant 0$, $\alpha^*_{2i} = (t_\alpha)^i$ *and* $\alpha^*_{2i+1} = (t_\alpha)^i F_\alpha$. [Cf. (**3**), Proposition 11.2.]

The proof, which we omit, can be based on Theorem 8, or directly on formulae (12.2) and (12.3).

13. A 'naturalizing homotopy' for the cyclic resolution

Let $\alpha: Q \to Q'$ and $\beta: Q' \to \bar{Q}$ be as in § 11. Formulae (12.2) and (12.3) yield Q-maps $\alpha_\sharp: X \to X'$, $\beta_\sharp: X' \to \bar{X}$, and $(\beta\alpha)_\sharp: X \to \bar{X}$. By (**1**), the maps $(\beta\alpha)_\sharp$ and $\beta_\sharp \alpha_\sharp$ of the Q-projective complex X into the acyclic complex $\bar{X}$ are Q-homotopic. Recalling (11.4), (11.7), and (11.12), we define a specific Q-homotopy $s: X \to \bar{X}$ by:

$$s_{2i}(1) = z_{\beta\alpha}\sigma_i \bar{M}_{\beta\alpha}; \qquad s_{2i+1}(1) = \sigma_{i+1} \quad (i \geqslant 0). \tag{13.1}$$

THEOREM 10. $sd + \bar{d}s = \beta_\sharp \alpha_\sharp - (\beta\alpha)_\sharp$.

The proof, which we omit, is based on Theorem 7 (§ 11) and [11.13].

14. Calculation of $\gamma: H_0^{2i+1}(\bar{Q}, G) \to H^1[P, H^{2i}(Q, G)]$

Throughout the rest of the paper, P, $\beta: Q \to \bar{Q}$, Q_0, and G will be as at the end of § 5. We assume further that Q is finite cyclic, of order n, and that P is cyclic. Let $m = 0$ if P is infinite; otherwise let m denote the order of P. Choose fixed generators $\mathbf{p}$, $\mathbf{q}$ of P and Q respectively, and let

$$\bar{\mathbf{q}} = \beta\mathbf{q}. \tag{14.1}$$

We shall apply the notations of §§ 10–13 to the sequence of homomorphisms

$$Q \xrightarrow{\mathbf{p}_Q} Q \xrightarrow{\beta} \bar{Q}. \tag{14.2}$$

Since $t_\beta = 1$, $u_{\mathbf{p}_Q} = t_{\mathbf{p}_Q}$, $\bar{F}_\beta = 1$, and $\beta\mathbf{p}_Q = \beta$, we shall use the following abbreviations:

$$t = t_{\mathbf{p}_Q} = u_{\mathbf{p}_Q}; \qquad u = u_\beta; \qquad F = F_{\mathbf{p}_Q}. \tag{14.3}$$

Notice that u is the order of Q_0, and that $c\bar{n} = t - 1$.

Applying Theorem 8 (§ 12) to $\beta: Q \to \bar{Q}$, we obtain

THEOREM 11 (Part 1). *The isomorphism*

$$H^{2i+1}(\bar{Q}, G) \approx \bar{N}^{-1}(0)/\bar{D}(G)$$

induces

$$H_0^{2i+1}(\bar{Q}, G) \approx A_{2i+1}/\bar{D}(G), \tag{14.4}$$

where A_{2i+1} denotes the group of all $g \in G$ such that $\bar{N}g = 0$ and $u^i g \in D(G)$.

Consider $h \in H^{2i}(Q, G)$. We recall from § 9 [cf. (3.15)] the convention $ph = hp^{-1}$. On the other hand, $(p_Q)^* h$ in the sense of § 12 means hp. Therefore Theorem 9 (§ 12) yields

$$\mathbf{p}^{-1}h = (\mathbf{p}_Q)^* h = t^i h \quad \text{for } h \in H^{2i}(Q, G). \tag{14.5}$$

THEOREM 11 (Part 2). *The combination of (a) formula (10.4) applied to the group P with generator $\mathbf{p}^{-1}$, (b) formula (10.4) applied to the group Q with generator $\mathbf{q}$, and (c) an appropriate Noether isomorphism, yields*

$$H^1[P, H^{2i}(Q, G)] \approx B_{2i}/C_{2i}, \tag{14.6}$$

where B_{2i} denotes the group of all $g \in G$ such that

$$Dg = 0, \qquad \sum_{j=0}^{m-1} t^{ij} g \in N(G)$$

and C_{2i} denotes the subgroup $N(G) + (t^i - 1)[D^{-1}(0)]$ of G.

We now define

$$\tau_i = \sum_{j=0}^{i-1} t^j. \tag{14.7}$$

If $g \in A_{2i+1}$, then $u^i g \in D(G)$. Hence the symbol $c\tau_i \bar{N} D^{-1} u^i$ describes, in an obvious manner, a multivalued function $A_{2i+1} \to G$. It may be verified that this induces a single-valued homomorphism with the domain and range indicated in diagram (14.8).

THEOREM 11 (Part 3). *The diagram*

$$\begin{array}{ccc} H_0^{2i+1}(\bar{Q}, G) & \approx & A_{2i+1}/\bar{D}(G) \\ \downarrow \gamma & & \downarrow (c\tau_i \bar{N} D^{-1} u^i) \\ H^1[P, H^{2i}(Q, G)] & \approx & B_{2i}/C_{2i} \end{array} \tag{14.8}$$

is commutative.

The proof, which we omit, is based mainly on Theorems 7 and 10 (§§ 11, 13).

COROLLARY. *Suppose that*

$$\gamma\colon H_0^3(\bar{Q}, G) \to H^1[P, H^2(Q, G)]$$

is not zero. Then: $uA_3 \neq 0$, $c\bar{N}(G) \neq 0$; none of the groups

$$\bar{N}(G), \quad D(G), \quad \bar{N}^{-1}(0), \quad D^{-1}(0)$$

is equal to G or zero; the inclusions

$$0 \subset D(G) \subset \bar{N}^{-1}(0) \subset G$$

are proper; the order of G, if finite, is the product of at least three primes (not necessarily distinct), and hence is at least 8; none of the following three formulae

$$\mathbf{q}g = g, \qquad \mathbf{q}g = -g, \qquad \mathbf{q}g = 2g$$

holds for all $g \in G$: not all elements of $\bar{N}^{-1}(0)$ are of order 2.

15. An example

Let P, Q, G be cyclic of order 2, 8, 8 respectively; let $\mathbf{pq} = \mathbf{q}^5$, and $\mathbf{q}g = 5g$ for all $g \in G$ let P operate trivially on G. By (10.4), $H^2(Q, G)$ is cyclic of order 4; hence, by (14.5), the induced operation of P on $H^2(Q, G)$ is trivial. It now follows by (10.4) that $H^1[P, H^2(Q, G)]$ is of order 2.

Choose a generator $\mathbf{h}$ of $H^2(Q, G)$, and a non-empty set D on which $H^2(Q, G)$ operates simply transitively. Define an operation of P on D by

$$\mathbf{p}d = 2\mathbf{h}+d \quad \text{for all } d \in D. \tag{15.1}$$

It is easily verified that D is now a distributive space over $[P, H^2(Q, G)]$. Clearly no point in D is left fixed by all of P; hence the deviation of D is the non-zero element of $H^1[P, H^2(Q, G)]$. From Theorem 11 (§ 14) it follows easily that

$$\gamma\colon H_0^3(\bar{Q}, G) \to H^1[P, H^2(Q, G)]$$

is an isomorphism of one group of order 2 onto another. Therefore, by Theorem 5 (§ 7), D is isomorphic to the distributive space of some Q-kernel $\theta\colon Q \to O(K)$, with P operating trivially on K. But inspection of (15.1) reveals that D contains exactly two orbits under P, while $H^2(Q, G)$ contains four. Hence $\theta\colon Q \to O(K)$ is a counter-example to Lemma 5 of (**2**).

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COMMUTATIVE RINGS OF LINEAR TRANSFORMATIONS AND INFINITE MATRICES

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1. Introduction

LET $\mathfrak{R}$ be an infinite-dimensional left vector space over a 'sfield' Δ, and let $\mathfrak{R}'$ be a right vector space over Δ [(**11**), 6]. Suppose that $\mathfrak{R}$ and $\mathfrak{R}'$ are dual spaces relative to a non-degenerate bilinear form (x, y) defined on $\mathfrak{R} \times \mathfrak{R}'$ to Δ [(**11**), 253]. Let $\mathfrak{L}(\mathfrak{R})$ be the ring of all linear transformations in $\mathfrak{R}$, and let $\mathfrak{L}(\mathfrak{R}'/\mathfrak{R})$ be the ring of all linear transformations in $\mathfrak{R}$ which have transposes in $\mathfrak{R}'$, the product AB of two transformations A and B being defined by $x(AB) = (xA)B$ $(x \in \mathfrak{R})$.

I shall consider commutative subrings of the ring $\mathfrak{L}(\mathfrak{R}'/\mathfrak{R})$, and from the results established I shall deduce, as particular cases, analogous results on commutative rings of infinite matrices.

The notation $\mathfrak{R}^*$ for the space of linear functions on a vector space $\mathfrak{R}$ (conjugate space of R) [(**11**), 52; (**7**), 42], and α^* for the Köthe–Toeplitz dual space of a sequence space α [(**8**), 275] are both widely used, and I have retained them.

The space of all $y \in \mathfrak{R}'$ such that $(x, y) = 0$ for every x in a subspace M of $\mathfrak{R}$ is called the *subspace incident* to M and is denoted by $j(M)$. Similarly $j(N)$ will denote the subspace of $\mathfrak{R}$ incident to a subspace N of $\mathfrak{R}'$. It is known that $j\{j(M)\}$ is the closure of M in the $\mathfrak{R}'$-topology of $\mathfrak{R}$ and, in particular, that M is closed if and only if $j\{j(M)\} = M$ [(**11**), 255].

In § 3 it is proved that with every proper closed subspace M $(M \neq 0)$ of $\mathfrak{R}$ we can associate a maximal commutative subring of $\mathfrak{L}(\mathfrak{R}'/\mathfrak{R})$.

Let Φ be the centre of Δ. The linear scalar transformations in $\mathfrak{R}$ are determined by the elements of Φ [(**11**), 35], and I shall denote the scalar transformation which corresponds to an element $a \in \Phi$ by a [(**5**), 207]. A transformation $T \in \mathfrak{L}(\mathfrak{R}'/\mathfrak{R})$ belongs to the commutative subring $\Phi(M)$ associated with the subspace M if and only if T induces a linear scalar transformation a in M and the transpose T' induces a in $j(M)$.

The rank $\rho(A)$ of a linear transformation A is the dimensionality of the image space $\mathfrak{R}A$ [(**11**), 256]. The set of all transformations of finite rank in $\mathfrak{L}(\mathfrak{R})$ will be denoted by $\mathfrak{F}$. I shall show that, when M has finite dimension or finite codimension [(**7**), 37], then the Jacobson radical $\theta(M)$ of $\Phi(M)$ [(**12**, 303; (**15**), 132] is $\mathfrak{F} \cap \Phi(M)$, but, when $\mathfrak{R}$ and $\mathfrak{R}'$ have

enumerable bases and M has infinite dimension and infinite codimension, then $\theta(M)$ contains transformations which are not of finite rank. In all cases, $\theta(M)$ is the maximal nilpotent ideal of $\Phi(M)$.

The set of all transformations in $\mathfrak{L}(\mathfrak{R}'/\mathfrak{R})$ which commute with a subset S of $\mathfrak{L}(\mathfrak{R}'/\mathfrak{R})$ is denoted by $\mathfrak{C}(S)$.

Let $\mathfrak{L}_M(\mathfrak{R}'/\mathfrak{R})$ be the ring of all transformations in $\mathfrak{L}(\mathfrak{R}'/\mathfrak{R})$ which leave a closed subspace M ($M \neq 0$) of $\mathfrak{R}$ invariant. It will be proved that, if $\mathfrak{B}$ is a proper two-sided ideal in $\mathfrak{L}_M(\mathfrak{R}'/\mathfrak{R})$, then $\mathfrak{C}(\mathfrak{B})$ is commutative, and that, if M has finite dimension or finite codimension, then $\mathfrak{C}(\mathfrak{B})$ is a maximal commutative subring of $\mathfrak{L}(\mathfrak{R}'/\mathfrak{R})$ if and only if $\mathfrak{B}$ is the Jacobson radical of $\Phi(M)$.

Some properties of infinite matrices which induce transformations of finite rank in a sequence space are given in § 4.

Little is known on the subject of commutativity of infinite matrices [(**8**), 100] and some of the results known involve matrices which are the transform of a diagonal matrix [(**8**), 101]. In a previous paper [(**3**), 225] I have studied abelian groups of infinite matrices, the group operation being matrix multiplication; commutativity in this case was established by the method employed in Theorem 2 for classes of transformations on an abstract vector space.

In § 5 we apply the general results previously established to rings of matrices. In order to do this, we identify rings of the type $\Sigma_\beta(\alpha)$ [(**9**), 307, 311] with rings of linear transformations in the sequence space α which have transposes in a dual space β. When α contains ϕ and $\Sigma(\alpha)$ is a ring, we can identify the ring $\Sigma(\alpha)$ with the ring of linear transformations in α with transposes in α^*, the Köthe–Toeplitz dual space of α.

Closed subspaces of a sequence space α in the α^*-topology of α have been introduced by G. Köthe in the theory of linear equations in infinitely many unknowns [(**13**), 203]. It has been proved by Köthe that, when α is a separable Banach space in which convergence and limit coincides with strong projective convergence and limit [(**8**), 302], then closure in the α^*-topology of α coincides with closure in the Banach space α [(**13**), 203]. This is the case for $\alpha = \sigma_r$ ($r \geqslant 1$).

In § 5 it is proved that the set of all matrices in $\Sigma_\beta(\alpha)$ which are the transform of a diagonal matrix in $\Sigma_\beta(\alpha)$ by a fixed matrix in $\Sigma_\beta(\alpha)$ is a maximal commutative subring of $\Sigma_\beta(\alpha)$. Evidently every maximal commutative subring of $\Sigma_\beta(\alpha)$ contains the ring of scalar matrices, and it will be proved that the intersection of a maximal commutative subring associated with a closed subspace and a maximal commutative subring of the diagonal type is the ring of scalar matrices.

Finally we consider subrings associated with a space λ of finite dimension, the space λ being a subspace of a perfect space μ [(**8**), 275] which is itself a proper subspace of a perfect space α. It will be proved that the maximal commutative subring of $\Sigma(\alpha)$ associated with λ is a subring of the maximal commutative subring of $\Sigma(\mu)$ associated with λ, but the maximal property does not hold in the ring $\Sigma(\mu)$.

Some examples of maximal commutative subrings in the rings of K_r, K_c, K, and Hilbert matrices are given: these commutative subrings do not contain any non-scalar matrix which is the transform of a diagonal matrix by a matrix in the corresponding ring.

2. Definitions and notation

The vector spaces $\mathfrak{R}$ and $\mathfrak{R}'$ will remain fixed throughout § 3 and § 4, but we shall consider subspaces of various types. Suppose M and N subspaces of $\mathfrak{R}$ and $\mathfrak{R}'$ (M, $N \neq 0$) and let $\Phi(N/M)$ denote the ring of all transformations $T \in \mathfrak{L}(\mathfrak{R}'/\mathfrak{R})$ to which correspond an element $a \in \Phi$ such that (i) $xT = ax$ for all $x \in M$, (ii) $yT' = ya$ for all $y \in N$. The mapping $T \to a$ of $\Phi(N/M)$ on Φ will be denoted by s, and I shall write $T^s = a$. We observe that, if $u_i \in j(N)$, $v_i \in j(M)$ ($i = 1, 2, \ldots, n$), and $a \in \Phi$, then the transformation A defined by

$$xA = ax + \sum_{i=1}^{n} (x, v_i)u_i$$

belongs to $\Phi(N/M)$.

We observe [(**16**), 158] that, if $T^s = a$ and $x \in M$, $y \in \mathfrak{R}'$, then

$$0 = (xT - ax, y) = (x, yT' - ya),$$

and hence
$$\mathfrak{R}'(T' - a) \subseteq j(M). \tag{2.1}$$

Similarly
$$\mathfrak{R}(T - a) \subseteq j(N). \tag{2.2}$$

If $x \in j\{j(M)\}$ and $y \in \mathfrak{R}'$, we have

$$(xT - ax, y) = (x, yT' - ya) = 0,$$

by (2.1), and hence $xT = ax$. Similarly $yT' = ya$ for every $y \in j\{j(N)\}$, and, since $j\{j(M)\} \supseteq M$, it follows from the definition above that

$$\Phi[j\{j(N)\}/j\{j(M)\}] = \Phi(N/M).$$

We shall therefore consider closed subspaces only. If $M = \mathfrak{R}$ or $N = \mathfrak{R}'$ $\Phi(N/M)$ is the ring of linear scalar transformations and I shall exclude these cases.

If N is a closed subspace of finite codimension n in $\mathfrak{R}'$, then $j(N)$ has dimension n [(**14**), 164], and it follows from (2.2) that $T - a$ has finite

rank. Similarly $T-a$ is of finite rank if M is a closed subspace of finite codimension in $\mathfrak{R}$.

Let 1 denote the identity element of the 'sfield' Δ. In order to describe transformations belonging to some rings of the type $\Phi(N/M)$, I shall use the following definitions:

DEFINITION 1. A subspace of codimension 1 in a vector space is called a *hyperplane* [(**7**), 38].

DEFINITION 2. If $u \in \mathfrak{R}, f \in \mathfrak{R}^*$, and $\langle u, f \rangle = 0$, the transformation A defined by $xA = x + \langle x, f \rangle u$, $x \in \mathfrak{R}$, is called a *transvection* [(**10**), 10].

Evidently $(A-1)^2 = 0$.

DEFINITION 3. If the characteristic of $\Delta \neq 2$, a linear transformation T $(\neq 1)$ such that $(T-1)^2 = 0$ is called a *transformation of class* 2 [(**5**), 203, 207].

Thus, if the characteristic of $\Delta \neq 2$, a linear transformation B $(\neq 1)$ is a transvection if and only if B is of class 2 and $\rho(B-1) = 1$.

The dual space $\mathfrak{R}'$ of $\mathfrak{R}$ is isomorphic to a subspace $\mathfrak{R}_1$ of $\mathfrak{R}^*$ under the correspondence $v \leftrightarrow g$, where $(x, v) = \langle x, g \rangle$ for every $x \in \mathfrak{R}$ [(**11**), 253]. If the transvection A defined above belongs to $\mathfrak{L}(\mathfrak{R}'/\mathfrak{R})$ and $u \neq 0$, then $f \in \mathfrak{R}_1$, and there is a vector $v' \in \mathfrak{R}'$ such that $xA = x + (x, v')u$.

3. Commutative subrings

It is to be understood throughout § 3 and § 4 that M and N are proper closed subspaces of $\mathfrak{R}$ and $\mathfrak{R}'$ and that $M \neq 0$, $N \neq 0$.

THEOREM 1. $\mathfrak{C}[\Phi(N/M)] = \Phi\{j(M)/j(N)\}$.

Suppose that $T \in \Phi(N/M)$ and let $T^s = b$. Suppose that

$$A \in \Phi\{j(M)/j(N)\}$$

and let a be the element of Φ which corresponds to A. From (2.2) we have $R(A-a) \subseteq M$, and hence $(A-a)(T-b) = 0$. Similarly

$$(T-b)(A-a) = 0,$$

and it follows that $AT = TA$.

Suppose that $u \in j(N)$ and $v \in j(M)$ $(u \neq 0, v \neq 0)$ and let C be the transformation defined in $\mathfrak{R}$ by $xC = (x, v)u$. Then $C \in \Phi(N/M)$, and, if $B \in \mathfrak{C}[\Phi(N/M)]$, we have

$$(x, v)uB = (xB, v)u = (x, vB')u. \tag{3.1}$$

There is a point $x' \in \mathfrak{R}$ such that $(x', v) = 1$; fix v and vary u, and we

obtain $uB = cu$ for every $u \in j(N)$, where $c \in \Phi$, since B is linear. From (3.1) we obtain

$$(x, vc)u = (x, vB')u$$

for every $x \in \mathfrak{R}$, and this implies that $vc = vB'$. Hence $B \in \Phi\{j(M)/j(N)\}$. This proves the theorem.

COROLLARY. *$\Phi(N/M)$ is commutative if and only if*

$$\Phi(N/M) \subseteq \Phi\{j(M)/j(N)\}.$$

THEOREM 2. *$\Phi(N/M)$ is commutative if $M \supseteq j(N)$.*

For $M \supseteq j(N)$ implies $N \supseteq j(M)$, and the result follows from the definition and the corollary to Theorem 1.

The space generated by a set V of vectors will be denoted by $[V]$.

THEOREM 3. *$\Phi(N/M)$ is commutative if M and N are closed hyperplanes and Δ is commutative.*

In this case the spaces $j(M)$ and $j(N)$ have dimension 1. Let $j(M) = [v]$, $j(N) = [u]$, and let A be the transformation defined by $xA = (x, v)u$, $x \in \mathfrak{R}$. If $T \in \Phi(N/M)$ and $T^s = a$, we have, by (2.2), $x(T-a) = \langle x, f \rangle u$, where $f \in \mathfrak{R}^*$. Since T has a transpose in $\mathfrak{R}'$, it follows that there is a point $v' \in \mathfrak{R}'$ such that

$$\langle x, f \rangle = (x, v').$$

If $y \in \mathfrak{R}'$ we have

$$y(T'-a) = v'(u, y),$$

and it follows from (2.1) that $v' = vb$, where $b \in \Delta$. Hence

$$x(T-a) = b(x, v)u, \qquad T = a+bA.$$

The commutativity follows.

If $(u, v) \neq 0$, we can choose u and v such that $(u, v) = 1$, and then A is an idempotent.

Suppose that $(u, v) = 0$, and let $B = 1+A$; then B is a transvection, and $\Phi(N/M)$ is the ring of all transformations $c+dB$, where c and d are elements of Φ.

THEOREM 4. *If $\Phi(N/M)$ is commutative, then either* (i) *$M \supseteq j(N)$, or* (ii) *Δ is commutative and M and N are hyperplanes.*

Suppose that (i) does not hold; then there are vectors $u_1 \in j(N)$ and $v_1 \in j(M)$ such that $(u_1, v_1) \neq 0$. Hence, if $a \neq 0$ belongs to Δ, there are vectors $u \in j(N)$ and $v \in j(M)$ such that $(u, v) = a$. The transformation A defined on $\mathfrak{R}$ by $xA = (x, v)u$ belongs to $\Phi(N/M)$, and hence, by the corollary to Theorem 1, we have $A \in \Phi\{j(M)/j(N)\}$. From the definitions, there is an element $b \in \Phi$ such that $xA = bx$ for $x \in j(N)$, and we have $bu = uA = au$. Hence $a = b$ and $\Delta = \Phi$. If $x \in j(N)$, we have

$$ax = (x, v)u \quad \text{and} \quad x = a^{-1}(x, v)u.$$

Thus $j(N) = [u]$ and similarly $j(M) = [v]$. It follows that M and N are hyperplanes.

THEOREM 5. *The ring $\Phi(N/M)$ is a maximal commutative subring of $\mathfrak{L}(\mathfrak{R}'/\mathfrak{R})$ if and only if $N = j(M)$.*

Suppose that $N = j(M)$; then $M = j(N)$ and $\mathfrak{C}[\Phi(N/M)] = \Phi(N/M)$, by Theorem 1. Hence $\Phi(N/M)$ is a maximal commutative subring of $\mathfrak{L}(\mathfrak{R}'/\mathfrak{R})$.

Conversely, suppose that $\Phi(N/M)$ is a maximal commutative subring of $\mathfrak{L}(\mathfrak{R}'/\mathfrak{R})$. By the corollary to Theorem 1, we have

$$\Phi(N/M) \subseteq \Phi\{j(M)/j(N)\}.$$

It follows from Theorem 1 that, if $T \in \Phi\{j(M)/j(N)\}$, the ring generated by T and $\Phi(N/M)$ is commutative, and from the maximal property we obtain $T \in \Phi(N/M)$. Hence

$$\Phi(N/M) = \Phi\{j(M)/j(N)\}.$$

It follows from Theorem 4 that either (i) $M \supseteq j(N)$ or (ii) M and N are hyperplanes. The ring $\Phi\{j(M)/j(N)\}$ is commutative, and hence either (iii) $j(N) \supseteq M$, or (iv) $j(M)$ and $j(N)$ are hyperplanes. If M and N are hyperplanes, neither (iii) nor (iv) can hold. If $j(M)$ and $j(N)$ are hyperplanes, neither (i) nor (ii) can hold. It follows from (i) and (iii) that $M = j(N)$.

The maximal commutative subring $\Phi\{j(M)/M\}$ of $\mathfrak{L}(\mathfrak{R}'/\mathfrak{R})$ will be denoted by $\Phi(M)$ and the Jacobson radical of $\Phi(M)$ will be denoted by $\theta(M)$.

THEOREM 6. $\{\theta(M)\}^2 = 0$.

Suppose $T_i \in \Phi(M)$ $(i = 1, 2)$ and let $T_i^s = a_i$. By (2.2) we have

$$\mathfrak{R}(T_2 - a_2) \subseteq M,$$

and hence

$$(T_1 - a_1)(T_2 - a_2) = 0.$$

We have $(T_1 - a_1)^2 = 0$ and, if $a_1 \neq 0$, it follows that T_1 has an inverse in the ring $\Phi(M)$. If $a_1 = 0$, T_1 is nilpotent. It follows that, if the characteristic of $\Delta \neq 2$, every transformation in $\Phi(M)$ which is non-scalar and non-nilpotent can be expressed uniquely as a product of a scalar transformation and a transformation of class 2. If T_1 and T_2 are nilpotent, then $T_1 T_2 = 0$. Hence $\theta(M)$ is the maximum nilpotent ideal of the ring $\Phi(M)$ and $\{\theta(M)\}^2 = 0$.

When we identify the linear scalar transformations in R with elements of Φ and observe that

$$T_1 = (T_1 - a_1) + a_1,$$

where $T_1 - a_1$ is nilpotent, it follows that $\Phi(M)$ is the direct sum of the field Φ and the Jacobson radical $\theta(M)$. I use the notation

$$\Phi(M) = \Phi \oplus \theta(M)$$

to denote the direct sum. Evidently

$$\mathfrak{C}[\theta(M)] = \Phi(M).$$

THEOREM 7. *The quotient ring $\Phi(M)/\theta(M)$ is isomorphic to Φ.*

Suppose that $T \in \Phi(M)$ and let $T^s = a$. The mapping $T \to a$ of $\Phi(M)$ on Φ is a homomorphism with kernel $\theta(M)$. The result follows [(**6**), 355].

THEOREM 8. (i) *If M has finite dimension or is a closed subspace of finite codimension in $\mathfrak{R}$, then $\theta(M) = \mathfrak{F} \cap \Phi(M)$.*

(ii) *If $\mathfrak{R}$ and $\mathfrak{R}'$ have enumerable bases and M is a closed subspace of infinite dimension and of infinite codimension in $\mathfrak{R}$, then $\theta(M)$ contains transformations which are not of finite rank.*

(i) In each case it follows from (2.1) or (2.2) that $\theta(M) \subseteq \mathfrak{F} \cap \Phi(M)$. Transformations in the ring $\mathfrak{F} \cap \Phi(M)$ have no inverse, and it follows from the proof of Theorem 6 that $\mathfrak{F} \cap \Phi(M) \subseteq \theta(M)$. This proves (i).

(ii) There is a subspace P of $\mathfrak{R}$ such that $\mathfrak{R} = M \oplus P$, and a subspace S of $\mathfrak{R}'$ such that $\mathfrak{R}' = j(M) \oplus S$. The spaces M and S are dual spaces relative to (x, y) $(x \in M,\ y \in S)$; for, if $x_1 \in M$ and $(x_1, y) = 0$ for every $y \in S$, it follows that $(x_1, y) = 0$ for every $y \in \mathfrak{R}'$ and hence $x_1 = 0$. If $y_1 \in S$ and $(x, y_1) = 0$ for every $x \in M$, then $y_1 \in j(M) \cap S$, and hence $y_1 = 0$. The spaces M and S have infinite dimension and have enumerable bases. Hence there is a basis (e_i) $(i = 1, 2,...)$ for M which has a complementary basis (f_i) $(i = 1, 2,...)$ in S in the sense that

$$(e_i, f_j) = \delta_{ij} \quad (i, j = 1, 2,...),$$

where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$ [(**11**), 253]. Similarly P and $j(M)$ are dual spaces and there is a basis (e_i') $(i = 1, 2,...)$ for P which has a complementary basis (f_i') $(i = 1, 2,...)$ in $j(M)$. There is a linear transformation T in $\mathfrak{R}$ such that $e_i T = 0$ and $e_i' T = e_i$ $(i = 1, 2,...)$ [(**11**), 243]. There is a linear transformation X in $\mathfrak{R}'$ such that $f_i' X = 0$ and $f_i X = f_i'$ $(i = 1, 2,...)$. Suppose that $x \in \mathfrak{R}$ and $y \in \mathfrak{R}'$; then

$$x = \sum_{i=1}^{n} x_i e_i + \sum_{i=1}^{n'} x_i' e_i',$$

where x_i, x_i' are in Δ, and

$$y = \sum_{i=1}^{m} f_i y_i + \sum_{i=1}^{m'} f_i' y_i',$$

where y_i, y_i' are in Δ. We have

$$xT = \sum_{i=1}^{n} x_i' e_i, \qquad yX = \sum_{i=1}^{m} f_i' y_i.$$

Since $(e_i, f_j') = 0$ $(i, j = 1, 2, \ldots)$, it follows that

$$(xT, y) = \sum x_i' y_i = (x, yX).$$

Hence $T' = X$, and, since $T^s = 0$, we have $T \in \theta(M)$. The image space of T is M, and T is not of finite rank.

THEOREM 9. *If $\mathfrak{B}$ is a proper two-sided ideal in $\mathfrak{L}_M(\mathfrak{R}'/\mathfrak{R})$, then* (i) *$\mathfrak{B}$ contains $\mathfrak{F} \cap \Phi(M)$*, (ii) *$\mathfrak{C}(\mathfrak{B})$ is a commutative ring.*

The ring $\mathfrak{L}_M(\mathfrak{R}'/\mathfrak{R})$ induces a dense ring of linear transformations on the space M, and hence $\mathfrak{B}$ contains all transformations of finite rank in $\mathfrak{L}_M(\mathfrak{R}'/\mathfrak{R})$ [(**11**), 261]. This proves (i). The result (ii) is deduced from the proof of Theorem 1. We take $N = j(M)$, so that $j(N) = M$, and suppose $B \in \mathfrak{C}[\mathfrak{B}]$. It follows as in the proof of Theorem 1 that $B \in \Phi(M)$, and hence $\mathfrak{C}[\mathfrak{B}] \subseteq \Phi(M)$.

THEOREM 10. *If $\mathfrak{B}$ is a proper two-sided ideal in the ring $\mathfrak{L}_M(\mathfrak{R}'/\mathfrak{R})$, then $\mathfrak{C}[\mathfrak{B}]$ is a maximal commutative subring of $\mathfrak{L}(\mathfrak{R}'/\mathfrak{R})$ if and only if $\mathfrak{B} \subseteq \theta(M)$.*

Suppose the maximal property to hold. Then $\mathfrak{C}[\mathfrak{B}] = \Phi(M)$, by Theorem 5 and Theorem 9. Since $\mathfrak{C}[\mathfrak{C}[\mathfrak{B}]] \supseteq \mathfrak{B}$, it follows that

$$\Phi(M) \supseteq \mathfrak{B}.$$

Every non-nilpotent transformation in $\Phi(M)$ has an inverse in the ring $\mathfrak{L}_M(\mathfrak{R}'/\mathfrak{R})$, by the proof of Theorem 6, and hence $\mathfrak{B} \subseteq \theta(M)$.

THEOREM 11. *If* (i) *M is a subspace of $\mathfrak{R}$ of finite dimension or a closed subspace of finite codimension in $\mathfrak{R}$*, (ii) *$\mathfrak{B}$ is a proper two-sided ideal in the ring $\mathfrak{L}_M(\mathfrak{R}'/\mathfrak{R})$, then the commutative ring $\mathfrak{C}(\mathfrak{B})$ is maximal in $\mathfrak{L}(\mathfrak{R}'/\mathfrak{R})$ if and only if $\mathfrak{B} = \theta(M)$.*

It follows from Theorem 8 and Theorem 9 that $\mathfrak{B} \supseteq \theta(M)$, and the result follows from Theorem 10.

4. Infinite matrices which induce transformations of finite rank

Let α^* denote the Köthe–Toeplitz dual space of a sequence space α. I use the notation

$$(x, y) = \sum_{i=1}^{\infty} x_i y_i$$

for the scalar product of $x = \{x_i\}$ and $y = \{y_i\}$. Suppose that $\alpha \supseteq \phi$ and $\phi \subseteq \beta \subseteq \alpha^*$ [(**8**), 273]; then α and β are dual spaces in the generalized

sense relative to (x, y) [(1), 503], and $\Sigma_\beta(\alpha)$ is a ring [(9), 311]. Every matrix $A \in \Sigma_\beta(\alpha)$ induces a transformation $T_A \in \mathfrak{L}(\beta/\alpha)$. I use the conventional notation Ax to denote the transform of a vector x by an infinite matrix A. Suppose that A and B are in $\Sigma_\beta(\alpha)$ and $x \in \alpha$, and let the product of the transformations induced by these matrices be defined as in § 1; then we have

$$xT_{AB} = ABx = xT_BT_A.$$

The mapping $A \to T_A$ is an anti-isomorphism of the ring $\Sigma_\beta(\alpha)$ on the ring $\mathfrak{L}(\beta/\alpha)$ [(6), 503].

THEOREM 12. *If $A \in \Sigma_\beta(\alpha)$, the transformation T_A is of finite rank m if and only if $A = UV$, where* (i) *the first m column vectors of the matrix U are linearly independent in α and all other column vectors of U are zero,* (ii) *the first m row vectors of V are linearly independent in β and all other row vectors of V are zero.*

Let $\rho(T_A) = m$. It follows from (4.1) that there are m linearly independent vectors $u^{(i)}$ $(i = 1, 2,..., m)$ in α and m linearly independent vectors $v^{(i)}$ $(i = 1, 2,..., m)$ in β such that

$$Ax = \sum_{i=1}^{m} (x, v^{(i)})u^{(i)}.$$

Let U be the matrix whose ith column vector is $u^{(i)}$ $(i = 1, 2,..., m)$, other column vectors being zero. Let V be the matrix whose ith row vector is

$$v^{(i)} \quad (i = 1, 2,..., m),$$

other row vectors being zero. Let $e^{(p)}$ $(p = 1, 2,...)$ denote the unit vector $e_p^{(p)} = 1$, $e_k^{(p)} = 0$ $(k \neq p)$. If $x \in \alpha$, we have

$$Vx = \sum_{i=1}^{m} (x, v^{(i)})e^{(i)}$$

and

$$UVx = \sum_{i=1}^{m} (x, v^{(i)})Ue^{(i)} = \sum_{i=1}^{m} (x, v^{(i)})u^{(i)}.$$

Hence $A = UV$. We observe that U and V are in $\Sigma_\beta(\alpha)$ [(9), 307]. The converse is evident.

Let I denote the unit matrix. It follows from § 2 that T_A is a transvection in α if and only if $A = I + U_1V_1$, where (i) the first column vector of U_1 is in α and all other column vectors of U_1 are zero, (ii) the first row vector of V_1 is in β and all other row vectors of V_1 are zero, (iii) $V_1U_1 = 0$.

THEOREM 13. *If $A \in \Sigma_\beta(\alpha)$ and $\rho(T_A) = m$, then A' maps every subspace of α^* which contains ϕ on the same m-dimensional subspace of β.*

It follows from Theorem 12 that $A' \in \Sigma(\alpha^*)$ and that $(Ax, y) = (x, A'y)$

for every $x \in \alpha$ and $y \in \alpha^*$. Hence T_A has a transpose in α^* induced by A'. The transpose of T_A in the conjugate space of α maps every total subspace of the conjugate space on the same subspace $[\phi_1, \phi_2, \ldots, \phi_m]$, where the ϕ_i are linearly independent [(**11**), 253]. The space β is isomorphic to a total subspace β_1 of the conjugate space of α. Since A' maps β into β, it follows that the transpose of T_A in the conjugate space of α maps β_1 into β_1. Hence $\phi_i \in \beta_1$, and there are points $v^{(i)} \in \beta$ which map on ϕ_i under the isomorphism. If λ is a subspace of α^* which contains ϕ, then λ is isomorphic to a total subspace λ_1 of the conjugate space of α. The transpose of T_A in the conjugate space maps λ_1 on $[\phi_1, \phi_2, \ldots, \phi_m]$, and hence A' maps λ on $[v^{(1)}, v^{(2)}, \ldots, v^{(m)}]$.

We observe that $A' \in \Sigma_\alpha(\beta)$ [(**9**), 308], and it follows that A maps every subspace of β^* which contains ϕ on the same m-dimensional subspace of α.

THEOREM 14. *If $A \in \Sigma_\beta(\alpha)$ and $\rho(T_A) = m$, then for all but a finite number of scalars k ($\neq 0$) the matrix $kI+A$ has a two-sided reciprocal in $\Sigma_\beta(\alpha)$. The reciprocal of $kI+A$ can be expressed in the form $k^{-1}I+B$, where $\rho(T_B) = m$.*

With the notation of Theorem 12, we have

$$Ax = \sum_{i=1}^{m} (x, v^{(i)}) u^{(i)} \quad (x \in \alpha).$$

Let $C = (c_{i,j})$ be the $m \times m$ matrix whose elements are defined by

$$c_{ij} = (u^{(j)}, v^{(i)}) \quad (i, j = 1, 2, \ldots, m).$$

Then

$$Au^{(j)} = \sum_{i=1}^{m} c_{i,j}\, u^{(i)}.$$

Let

$$\lambda = [u^{(1)}, u^{(2)}, \ldots, u^{(m)}].$$

Suppose that $x \in \lambda$, and let x_i ($i = 1, 2, \ldots, m$) be the coordinates of x relative to this basis. Then

$$x = \sum_{j=1}^{m} x_j\, u^{(j)}, \qquad Ax = \sum_{i=1}^{m} \Big(\sum_{j=1}^{m} c_{i,j}\, x_j \Big) u^{(i)}.$$

Hence C is the matrix of the linear transformation induced by T_A in λ relative to the given basis. Let I_m denote the unit $m \times m$ matrix, and choose $k \neq 0$ such that $\det(kI_m + C) \neq 0$. Suppose that $z \in \alpha$ and consider the equation

$$(kI+A)x = z.$$

Let $\alpha = \lambda \oplus \mu$, and let $z = z^{(1)} + z^{(2)}$, where $z^{(1)} \in \lambda$ and $z^{(2)} \in \mu$. Since A maps α on λ, we have

$$z^{(1)} - k^{-1} A z^{(2)} \in \lambda.$$

Let d_i $(i = 1, 2,..., m)$ be the coordinates of $z^{(1)}-k^{-1}Az^{(2)}$ relative to the basis $u^{(i)}$. There are complex numbers x_i $(i = 1, 2,..., m)$ such that

$$kx_i+\sum_{j=1}^{m} c_{i,j}x_j = d_i \quad (i = 1, 2,..., m).$$

Let $$x^{(1)} = x_1u^{(1)}+x_2u^{(2)}+...+x_mu^{(m)},$$

and let $x^{(2)} = k^{-1}z^{(2)}$. Then

$$(kI+A)x^{(1)} = z^{(1)}-k^{-1}Az^{(2)}, \qquad (kI+A)x^{(2)} = z^{(2)}+k^{-1}Az^{(2)}.$$

Hence $kI+A$ maps $x^{(1)}+x^{(2)}$ on z. It follows that $kI+A$ maps α on α, and similarly $kI+A'$ maps β on β. Hence $kI+A$ has a two-sided reciprocal in $\Sigma_\beta(\alpha)$ [(**4**), 505], and we express this reciprocal in the form $k^{-1}I+B$.

If $x \in j[v^{(1)}, v^{(2)},..., v^{(m)}]$, we have $Ax = 0$, by Theorem 12, and

$$x = (k^{-1}I+B)(kI+A)x = x+kBx,$$

and so $Bx = 0$. Thus B maps $j[v^{(1)}, v^{(2)},..., v^{(m)}]$ on 0, and this implies that B' maps β into $[v^{(1)}, v^{(2)},..., v^{(m)}]$. Hence $\rho(B) \leqslant \rho(A)$, and similarly $\rho(A) \leqslant \rho(B)$. This proves that $\rho(B) = m$.

5. Maximal commutative subrings of $\Sigma_\beta(\alpha)$

In § 5 the spaces on which the transformations are defined will vary, and we shall consider commutative rings which are subrings of more than one ring of the type $\Sigma_\beta(\alpha)$. In order to exhibit this property, a change in the notation is necessary. Let $\mathfrak{C}$ denote the complex field, and let $\lambda \neq 0$ be a proper closed subspace of α in the β-topology of α. Let $C_\lambda(\alpha/\beta)$ denote the ring of all $A \in \Sigma_\beta(\alpha)$ such that

(i) $Ax = ax$ for $x \in \lambda$,

(ii) $A'y = ay$ for $y \in j(\lambda)$, where $a \in \mathfrak{C}$.

It follows from Theorem 5 that $C_\lambda(\alpha/\beta)$ is a maximal commutative subring of $\Sigma_\beta(\alpha)$. The Jacobson radical of this ring will be denoted by $J_\lambda(\alpha/\beta)$.

Let 0_1 denote the space of all sequences $\{x_k\}$ such that $x_{2n-1} = 0$ $(n = 1, 2,...)$, and let 0_2 denote the space of all $\{x_k\}$ such that $x_{2n} = 0$ $(n = 1, 2,...)$ [(**8**), 273].

THEOREM 15. *If* (i) $\alpha \supseteq \phi$ *and* $\Sigma(\alpha)$ *is a ring,* (ii) S *is the set of all* $A \in \Sigma(\alpha)$ *such that* $a_{n,2k} = 0$ $(n, k = 1, 2,...)$ *and* $a_{2n-1,k} = 0$ $(n, k = 1, 2,...)$, *then the set of all matrices* $pI+A$, *where* $A \in S$ *and* $p \in \mathfrak{C}$, *is a maximal commutative subring of* $\Sigma(\alpha)$.

We take $\beta = \alpha^*$ and observe that $\Sigma_{\alpha^*}(\alpha) = \Sigma(\alpha)$ [(**9**), 309, 312]. Let $\lambda = 0_1 \cap \alpha$; then, if $y \in j(\lambda)$, we have $(e^{(2n)}, y) = y_{2n} = 0$. Evidently

$$j(\lambda) = 0_2 \cap \alpha^*,$$

and λ is closed. Hence $C_\lambda(\alpha/\alpha^*)$ is a maximal commutative subring of $\Sigma(\alpha)$, and it follows from the proof of Theorem 6 that every matrix in this subring is of the form $pI+A$, where $p \in \mathfrak{C}$ and $A \in J_\lambda(\alpha/\alpha^*)$. If $A \in J_\lambda(\alpha/\alpha^*)$, then $Ax = 0$ for $x \in \lambda$ and $A'y = 0$ for $y \in j(\lambda)$. It follows that row vectors of A are in $j(\lambda)$ and column vectors of A are in λ. This proves the theorem.

We obtain examples of maximal commutative subrings in the rings of (i) K-matrices, (ii) K_r-matrices, (iii) K_c-matrices, (iv) Hilbert matrices, (v) column-finite matrices, (vi) row-finite matrices, by suitably choosing the space α and applying Theorem 15 [(**9**), 298, 301, 324].

We shall require the following lemma:

LEMMA. *Suppose that* (i) α *and* μ *are perfect spaces and* μ *is a proper subspace of* α, (ii) λ *is a subspace of* μ. *Then* (*a*), *if* λ *is a closed subspace of* α *in the* α^**-topology of* α, *the space* λ *is a closed subspace of* μ *in the* μ^**-topology of* μ. (*b*), *if* λ *has finite codimension* m *in* μ, λ *is not a closed subspace of* α *in the* α^**-topology of* α.

(*a*) We have $\alpha^* \subset \mu^*$. Let $j_1(\lambda)$ be the subspace of α^* incident to λ and $j_2(\lambda)$ the subspace of μ^* incident to λ; then $j_1(\lambda) \subseteq j_2(\lambda)$. Hence the subspace of μ incident to $j_2(\lambda)$ is contained in the subspace of α incident to $j_1(\lambda)$, i.e. is contained in λ. This proves (*a*).

(*b*) Suppose that λ has codimension m in μ. The closure of the subspace λ of μ in the μ^*-topology of μ contains λ, and therefore has codimension $n \leqslant m$ in μ. It follows that

$$\dim j_2(\lambda) = n, \quad \text{and hence} \quad \dim j_1(\lambda) \leqslant n.$$

Thus the codimension of the closure of the subspace λ of α in the α^*-topology of α does not exceed m. Since the codimension of λ in α does exceed m, it follows that λ is not closed in α.

THEOREM 16. *If* λ *is a proper closed subspace of a perfect space* α *in the* α^**-topology of* α *and has finite codimension in* α, *then* $\alpha = \lambda^{**}$.

From $\alpha \supset \lambda$, it follows that $\alpha = \alpha^{**} \supseteq \lambda^{**}$ [(**8**), 275]. The space λ is not perfect; for, if λ is a perfect space, we have $\lambda \supseteq \phi$ [(**8**), 275], and hence $j(\lambda) = 0$ and $\lambda = \alpha$. Hence $\lambda^{**} \supset \lambda$. Suppose that $\alpha \supset \lambda^{**}$; then, by the lemma, λ is a closed subspace of λ^{**} in the λ^*-topology of λ^{**}. Since λ has finite codimension in α, it has finite codimension in λ^{**},

and it follows from the lemma that λ is not closed in the α^*-topology of α. The contradiction proves that $\alpha = \lambda^{**}$.

THEOREM 17. *If* (i) α *and* μ *are perfect spaces and* $\alpha \supset \mu$, (ii) λ *is a subspace of* μ $(\lambda \neq 0)$, (iii) λ *is a closed subspace of* α *in the* α^**-topology of* α, *then the maximal commutative subring* $C_\lambda(\alpha/\alpha^*)$ *of the ring* $\Sigma(\alpha)$ *is a subring of the maximal commutative subring* $C_\lambda(\mu/\mu^*)$ *of the ring* $\Sigma(\mu)$. *If* λ *has finite dimension,* $C_\lambda(\alpha/\alpha^*)$ *is a proper subring of* $C_\lambda(\mu/\mu^*)$.

Suppose that $A \in J_\lambda(\alpha/\alpha^*)$; then $A\alpha \subseteq \lambda \subset \mu$, by (2.2), and hence $A\mu \subseteq \mu$. It follows that $A \in \Sigma(\mu)$. We use the same notation as in the lemma. We have $Ax = 0$ for $x \in \lambda$ and $A'y = 0$ for $y \in j_1(\lambda)$. Hence the row vectors of A' belong to α [(**9**), 299] and are incident to $j_1(\lambda)$, and it follows from (iii) that the row vectors of A' are in λ. Hence $A'y = 0$ for $y \in j_2(\lambda)$, and from the definition we have $A \in J_\lambda(\mu/\mu^*)$. Since

$$C_\lambda(\alpha/\alpha^*) = \mathfrak{C} \oplus J_\lambda(\alpha/\alpha^*),$$

by the proof of Theorem 6, the result follows.

If $\dim \lambda = m$, $j_1(\lambda)$ has codimension m in α^*, and $j_2(\lambda)$ has codimension m in μ^*. Since α^* is a proper subspace of μ^*, it follows that $j_1(\lambda)$ is a proper subspace of $j_2(\lambda)$. Suppose $v = \{v_k\}$ is in $j_2(\lambda)$ and not in $j_1(\lambda)$; then v is in μ^* and not in α^*. Hence there is a point $z = \{z_k\} \in \alpha$ such that the series $\Sigma|z_k v_k|$ diverges. Suppose that

$$u = \{u_k\} \in \lambda \quad (u \neq 0),$$

and let $U = (u_{n,k})$, where

$$u_{n,1} = u_n \quad (n = 1, 2,\ldots), \qquad u_{n,k} = 0 \quad (k > 1).$$

Let $V = (v_{n,k})$, where

$$v_{1,k} = v_k \quad (k = 1, 2,\ldots), \qquad v_{n,k} = 0 \quad (n > 1).$$

The matrix $B = UV$ does not belong to $\Sigma(\alpha)$ since the series $\Sigma|z_k v_k|$ diverges. Since $u \in \lambda$ and $v \in j_2(\lambda)$, it follows that $Bx = 0$ for $x \in \lambda$, and $B'y = 0$ for $y \in j_2(\lambda)$. Evidently $B \in \Sigma(\mu)$, and hence $B \in J_\lambda(\mu/\mu^*)$. It follows that $C_\lambda(\alpha/\alpha^*)$ is a proper subring of $C_\lambda(\mu/\mu^*)$, and hence that the commutative ring $C_\lambda(\alpha/\alpha^*)$ is not maximal in $\Sigma(\mu)$.

Let $\mathfrak{D}$ be the ring of all diagonal matrices.

THEOREM 18. *If the matrix* $B \in \Sigma_\beta(\alpha)$ *has a two-sided reciprocal* $B^{-1} \in \Sigma_\beta(\alpha)$, *then the ring* $B^{-1}\{\mathfrak{D} \cap \Sigma_\beta(\alpha)\}B$ *is a maximal commutative subring of* $\Sigma_\beta(\alpha)$.

We prove the result first for the case $B = I$. Let $D^{(n)}$ $(n = 1, 2,\ldots)$ be the diagonal matrix with diagonal elements

$$d_k^{(n)} = k \quad (k = 1, 2,\ldots, n), \qquad d_k^{(n)} = 0 \quad (k > n).$$

Then $D^{(n)} \in \Sigma_\beta(\alpha)$. Suppose that $A \in \Sigma_\beta(\alpha)$ and

$$AD^{(n)} = D^{(n)}A \quad (n = 1, 2,...);$$

then

$$a_{ik}(d_k^{(n)} - d_i^{(n)}) = 0$$

and it follows that $a_{ik} = 0$ $(i \neq k)$. Hence the result holds for $B = I$. If $X \in \Sigma_\beta(\alpha)$ and

$$X(B^{-1}D^{(n)}B) = (B^{-1}D^{(n)}B)X \quad (n = 1, 2,...),$$

we have

$$(BXB^{-1})D^{(n)} = D^{(n)}(BXB^{-1}),$$

and therefore BXB^{-1} is a diagonal matrix. If $BXB^{-1} = D$, we have $X = B^{-1}DB$. This proves the theorem.

We proceed to prove that the intersection of two maximal commutative subrings, one of each of the classes referred to above, is the ring of scalar matrices.

THEOREM 19. *If B and B^{-1} are in $\Sigma_\beta(\alpha)$, then*

$$C_\lambda(\alpha/\beta) \cap B^{-1}\{\mathfrak{D} \cap \Sigma_\beta(\alpha)\}B$$

is the ring of scalar matrices.

Suppose that $D = (d_n) \in \Sigma_\beta(\alpha)$ and that $A = B^{-1}DB \in C_\lambda(\alpha/\beta)$. If $x \in \lambda$, we have $ax = Ax = B^{-1}DBx$, where $a \in \mathfrak{C}$. Hence $D(Bx) = a(Bx)$ and $d_n(Bx)_n = a(Bx)_n$ $(n = 1, 2,...)$. Let m be any fixed positive integer; if there is an $x \in \lambda$ such that $(Bx)_m \neq 0$, then $d_m = a$. If $(Bx)_m = 0$ for every $x \in \lambda$, and $b^{(m)}$ denotes the mth row vector of B, we have

$$(x, b^{(m)}) = 0 \quad (x \in \lambda).$$

Hence $b^{(m)} \in j(\lambda)$ and $A'b^{(m)} = ab^{(m)}$, from the definitions. From

$$(B^{-1})'B' = I,$$

we obtain $(B^{-1})'b^{(m)} = e^{(m)}$, and hence

$$ab^{(m)} = B'D(B^{-1})'b^{(m)} = B'De^{(m)} = d_m b^{(m)}.$$

Since $b^{(m)} \neq 0$, it follows that $d_m = a$ and $D = aI$. This proves the theorem.

6. Ideals in rings of matrices

The unit vectors $e^{(n)}$ $(n = 1, 2,...)$ form a basis for ϕ. Let $u = \{u_k\}$, where $u_k = 1$ $(k = 1, 2,...)$. The space C of stationary sequences [(**8**), 273] is the direct sum of its subspaces ϕ and $[u]$ by (**2**), and so C has an enumerable basis. Similarly the space D of alternating sequences has an enumerable basis. It follows from Theorem 8 that, if α is one of the spaces ϕ, C, or D, and if λ is a closed subspace of α in the ϕ-topology of α of infinite dimension and infinite codimension, then the Jacobson

radical of the commutative subring $C_\lambda(\alpha/\phi)$ of $\Sigma_\phi(\alpha)$ contains nilpotent matrices of index 2 [(**8**), 9] which induce transformations in α which are not of finite rank.

THEOREM 20. *Every proper two-sided ideal in the ring $\Sigma_\beta(\alpha)$ contains the ideal $\mathfrak{F} \cap \Sigma_\beta(\alpha)$.*

We identify the ring $\Sigma_\beta(\alpha)$ with the ring of linear transformations in α which have transposes in the dual space β. This is a dense ring of linear transformations and the result follows [(**11**), 261].

COROLLARY. *If $\alpha \supseteq \phi$ and $\Sigma(\alpha)$ is a ring, then every proper two-sided ideal in $\Sigma(\alpha)$ contains the ideal $\mathfrak{F} \cap \Sigma(\alpha)$.*

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ON THE CONTINUITY PROPERTIES OF VECTOR-VALUED FUNCTIONS OF BOUNDED VARIATION

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1. LET X be a Banach space over the complex field and let $x(\cdot)$ be a function from the real line to X with the property

$$[\mathrm{A}] \qquad \left\| \sum_{s=1}^{k} (x(t_{2s}) - x(t_{2s-1})) \right\| \leqslant K(a,b) < \infty$$

whenever $k \geqslant 1$ *and* $-\infty < a \leqslant t_1 < \ldots < t_{2k} \leqslant b < \infty$;

then we shall say that $x(\cdot)$ *is of weakly bounded variation* [in the sense of Gel'fand (**7**) and Dunford (**3**)]. It is known that $x(\cdot)$ has the property [A] if and only if the complex-valued function $x^*(x(\cdot))$ is of bounded variation in the usual sense on every compact interval of the real line for each choice of the bounded linear functional x^*. When X is a Hilbert space, then Edwards (**4**) has shown that [A] can also be expressed in terms of the covariance-function $\rho(s,t) \equiv (x(s), x(t))$. In the present paper we shall constantly make the assumptions [A] and

[B] *the space* X *is sequentially weakly complete,*

and we shall then show that the strong limits $x(t \pm 0)$ exist for all values of t, that $x(\cdot)$ is strongly continuous save at most on a countable t-set, that the range of $x(\cdot)$ is separable, and that (in case $x(\cdot)$ is continuous to the right) this function can be decomposed into a strongly continuous part and a step-function. These results were required in connexion with an investigation by Edwards and Moyal, partly reported in (**5**), into stochastic differential equations, and a number of further applications (to be reported elsewhere) have subsequently been noted by Edwards.

In the statistical applications X is normally a Hilbert space and [B] is then always satisfied; the Hilbert-space versions of our results (which can also be proved directly with the aid of covariance-function techniques) have applications to the expansion theorems of stochastic process theory (for an account of these see a forthcoming book by Moyal).

The assumption [B] will also hold whenever X is a reflexive space or an L-space.

Earlier contributions to this subject were made by Gel'fand (**7**) and by Alexiewicz (**1**). Certain of their results come close to ours, but by exploiting the properties of unconditionally convergent series in sequentially weakly complete spaces we have been able in one or two respects to assume less or prove more.†

We know of no detailed account of the work of Orlicz on unconditionally convergent series, although a summary of the results (without proofs) will be found at the end (pp. 240–1) of Banach's book (**2**). For the proofs of the theorems needed here we therefore refer the reader to a paper by Pettis (**8**). He will also find in (**6**) a substantial generalization by Edwards of the results now to be presented.

2. In what follows [A] and [B] will always be assumed; thus we can write, for any x^* in the adjoint space X^*,

$$x^*(x(t)) = \sum_{r=0}^{3} i^r f_r(t; x^*) \tag{1}$$

where, for each value of r, $f_r(\cdot\,; x^*)$ is a (finite) real-valued non-decreasing function of the real variable t. (The functions f_r are, of course, not uniquely determined by (1).) Accordingly the limits $f_r(t\pm 0; x^*)$ exist for each t and each x^*, and so the limits $x(t\pm 0)$ exist at least *weakly*. Now let $t_n \uparrow t$; then

$$\sum_{n=1}^{\infty} |x^*(x(t_{n+1})-x(t_n))| < \infty$$

for each $x^* \in X^*$ and so $\sum_{n=1}^{\infty} (x(t_{n+1})-x(t_n))$ is unconditionally convergent. It follows that $x(t_n)$ converges strongly (as $n \to \infty$) to a limit which can only be the weak limit $x(t-0)$. But the sequence $\{t_n\}$ was arbitrary, and so from this and a similar argument it follows that the *strong* limits $x(t\pm 0)$ exist for all t.

Next, let $u(t) \equiv x(t)-x(t-0)$ and $v(t) \equiv x(t+0)-x(t)$, and let $\{t_n\}$ be any sequence of distinct t-values interior to some compact t-interval. Then

$$\sum_{n=1}^{\infty} |x^*(u(t_n))| < \infty, \qquad \sum_{n=1}^{\infty} |x^*(v(t_n))| < \infty,$$

and so $\sum_{n=1}^{\infty} u(t_n)$ and $\sum_{n=1}^{\infty} v(t_n)$ are unconditionally convergent, and consequently $u(t_n)$ and $v(t_n)$ each tend strongly to zero when $n \to \infty$. It follows that for each value of $c > 0$ and for each compact interval J, each of the sets

$$\{t\colon \|u(t)\| \geqslant c\}, \qquad \{t\colon \|v(t)\| \geqslant c\},$$

† Thus Satz 4 (Anm. 1°) of § 6 of (**7**) and Theorem 7 of (**1**) are each weaker than our Theorem 1.

can have only finitely many points in common with J, and so $u(t) = v(t) = 0$ save on a countable t-set. We have now proved

THEOREM 1. *If* [A] *and* [B] *are satisfied, then* $x(\cdot)$ *is strongly continuous save at most on a countable set, and the strong limits* $x(t\pm 0)$ *exist at each exceptional point.*

3. If we put $\tilde{x}(t) \equiv x(t+0)$, we obtain a function $\tilde{x}(\cdot)$ which still satisfies [A] but which is strongly continuous to the right,

i.e. $$\tilde{x}(t+0) = \tilde{x}(t)$$

for all t. Let us suppose (throughout the rest of § 3) that this normalization has been effected, and write $y(t) \equiv x(t)-x(t-0)$ for the 'jump' of the normalized function at t. If $\{t_n\}$ is a sequence of distinct t-values interior to a compact interval, then $\sum |x^*(y(t_n))| < \infty$, and so $\sum y(t_n)$ will be unconditionally convergent. Thus, if τ_n $(n = 1, 2,...)$ is an enumeration of the points of discontinuity of the function $x(\cdot)$, then we can define

$$\left.\begin{aligned} z(t) &\equiv \sum_{0<\tau_n\leqslant t} y(\tau_n) \quad (t>0) \\ z(0) &\equiv 0 \\ z(t) &\equiv -\sum_{t<\tau_n\leqslant 0} y(\tau_n) \quad (t<0) \end{aligned}\right\}, \tag{2}$$

and $w(t) \equiv x(t)-z(t)$. It will be found that $z(\cdot)$ and $w(\cdot)$ both satisfy [A], so that the strong limits $z(t\pm 0)$ and $w(t\pm 0)$ exist. Some straightforward calculations now give

$$z(t-0) = z(t)-y(t), \qquad z(t+0) = z(t),$$

and so $w(t-0) = w(t) = w(t+0)$. We have therefore proved

THEOREM 2. *Let* [A] *and* [B] *be satisfied and suppose that* $x(\cdot)$ *is continuous to the right. Then*

$$x(t) = z(t)+w(t) \quad (-\infty < t < \infty),$$

where $z(\cdot)$ *and* $w(\cdot)$ *each satisfy* [A], $w(\cdot)$ *is strongly continuous, and* $z(\cdot)$ *is the step-function defined above at* (2).

4. On noting that $x(t)$ differs from $x(t+0)$ only on a countable set, and that $x(t+0)$ is the strong limit of $x(r)$ when $r \downarrow t$ through rational values, we obtain

THEOREM 3. *Every function of weakly bounded variation from the real line to a sequentially weakly complete space has a separable range.*

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ON THE CONTINUITY PROPERTIES OF FUNCTIONS SATISFYING A CONDITION OF SIRVINT'S

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1. In their study (**5**) of the continuity properties of vector-valued functions of bounded variation Kendall and Moyal proved a number of theorems subject to the condition that the function under consideration took its values in a weakly (sequentially) complete Banach space. In this paper it will be shown that this hypothesis of weak completeness can be replaced by a certain compactness condition. In the course of the discussion a number of related questions will also be considered.

If $x(\cdot)$ is a given function from the real line R to a complex Banach space X and if $-\infty < a < b < \infty$, then $E(a,b)$ will denote the set of all finite sums of the form

$$\sum_{i=1}^{n} \{x(t_{2i}) - x(t_{2i-1})\}$$

in which $n \geqslant 1$ and $a \leqslant t_1 < t_2 < \ldots < t_{2n} \leqslant b$. The class of all functions $x(\cdot)$ for which $E(a,b)$ is bounded will be denoted by $\mathscr{D}(a,b)$ and that for which $E(a,b)$ is conditionally weakly compact by $\mathscr{S}(a,b)$. These classes were introduced by Dunford (**2**) and Sirvint (**9**) respectively, and it is evident that

$$\mathscr{D}(a,b) \supseteq \mathscr{S}(a,b) \quad (-\infty < a < b < \infty).$$

Whereas Kendall and Moyal were concerned with functions of the class

$$\mathscr{D} \equiv \bigcap_{-\infty < a < b < \infty} \mathscr{D}(a,b),$$

in the present paper I shall consider the corresponding Sirvint class $\mathscr{S}$. We shall see that

$$\mathscr{D}(a,b) = \mathscr{S}(a,b) \quad (-\infty < a < b < \infty)$$

when the space X is weakly complete, and this equation relates the Kendall–Moyal theorems to those proved here.

The importance of the class $\mathscr{D}$ resides mainly in the fact, discovered by Dunford (**2**) and Gel'fand (**4**), that $x(\cdot) \in \mathscr{D}$ if and only if $x^*x(\cdot)$ is of bounded variation, in the ordinary sense, in each compact interval of R for each fixed element x^* of the adjoint space X^*. The Sirvint class $\mathscr{S}$ derives its importance from the fact that each function $x(\cdot)$ in $\mathscr{S}$ is

associated in a natural way† with a vector-valued measure defined on the ring $\mathscr{B}$ of real bounded Borel sets, and conversely.

2. The modifications of the method of Kendall and Moyal required for the proof of the continuity theorems of § 3 are made possible by the following theorem.

THEOREM 1. *Let X be a complex Banach space and let $\{x_n\}$ be an infinite sequence in X. Then a necessary and sufficient condition for the unconditional convergence of $\sum_1^\infty x_n$ is that the set*

$$E_0 \equiv \Big\{ \sum_{m=1}^{M} x_{n_m} : M \geqslant 1;\ 1 \leqslant n_1 < n_2 < \ldots < n_M < \infty \Big\}$$

be conditionally weakly compact.

If E_0 is conditionally weakly compact, then the set

$$E_1 \equiv \{x-y : x, y \in E_0 \cup (0)\}$$

is bounded: in other words, there is a finite constant A such that

$$\Big\| \sum_{n=1}^{N} \epsilon_n x_n \Big\| \leqslant A \quad \text{for } N = 1, 2, \ldots$$

whenever $\epsilon_n = 0, \pm 1$ (all n). If $x^* \in X^*$, it now follows, on writing

$$\epsilon_n \equiv \operatorname{sgn} \operatorname{re} x^*(x_n); \qquad \epsilon_n' \equiv \operatorname{sgn} \operatorname{im} x^*(x_n),$$

that

$$\begin{aligned} \sum_{n=1}^{N} |x^*(x_n)| &\leqslant \sum_{n=1}^{N} |\operatorname{re} x^*(x_n)| + \sum_{n=1}^{N} |\operatorname{im} x^*(x_n)| \\ &= \operatorname{re} x^*\Big(\sum_{n=1}^{N} \epsilon_n x_n \Big) + \operatorname{im} x^*\Big(\sum_{n=1}^{N} \epsilon_n' x_n \Big) \\ &\leqslant 2A\|x^*\| \quad (N = 1, 2, \ldots). \end{aligned}$$

Hence $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$ for each $x^* \in X^*$. Consequently, for each x^* and for each subseries $\sum_{m=1}^{\infty} x_{n_m}$,

$$\lim_{M\to\infty} x^*\Big(\sum_{m=1}^{M} x_{n_m} \Big) = \lim_{M\to\infty} \sum_{m=1}^{M} x^*(x_{n_m}) \equiv g(x^*)$$

exists, and, since E_0 is bounded, $g \in X^{**}$. But Eberlein (3) has shown that a set $E \subseteq X$ is conditionally weakly compact if and only if every sequence in E contains a subsequence weakly convergent to an element

† The relationship resembles that between ordinary real functions of bounded variation and signed measures defined on $\mathscr{B}$. For a precise statement see § 4.

of X. It therefore follows, since E_0 is conditionally weakly compact, that there exists a strictly increasing sequence $\{M_r\}$ of natural numbers and a vector $x_0 \in X$ such that

$$\text{weak} \lim_{r\to\infty} \sum_{m=1}^{M_r} x_{n_m} = x_0.$$

Evidently $g(x^*) = x^*(x_0)$ for all $x^* \in X^*$, and so

$$\lim_{M\to\infty} x^*\Big(\sum_{m=1}^{M} x_{n_m}\Big) = x^*(x_0)$$

for all x^*. The original series $\sum x_n$ thus has the property that each of its subseries is weakly convergent to an element of X; and so, by the Orlicz–Banach–Pettis theorem [see Pettis (**7**), Lemma 2.32, or Dunford (**2**), 322], $\sum x_n$ converges unconditionally.

Instead of now giving a direct proof of the necessity of the condition I shall, for the sake of completeness, prove the following somewhat stronger result. Here l will denote, as usual, the Banach space of infinite complex sequences

$$y \equiv (\eta_1, \eta_2, \ldots)$$

such that

$$\|y\| \equiv \sum_{n=1}^{\infty} |\eta_n| < \infty,$$

while m will denote the space of infinite bounded complex sequences

$$u \equiv (\alpha_1, \alpha_2, \ldots)$$

with the norm defined by

$$\|u\| = \sup_{1\le n<\infty} |\alpha_n|.$$

THEOREM 2. *A necessary and (trivially) sufficient condition for $\sum x_n$ to be unconditionally convergent is that $\sum \alpha_n x_n$ should converge strongly for every vector $u \equiv (\alpha_1, \alpha_2, \ldots) \in m$.*

It is then the case that $\sum \alpha_n x_n$ converges unconditionally for each u and that

$$Tu \equiv T(\alpha_1, \alpha_2, \ldots) = \sum_{n=1}^{\infty} \alpha_n x_n$$

defines a strongly (and hence weakly) compact operator $T: m \to X$.

This theorem, as can be seen from the proof, is little more than a restatement of some of the results of Pettis and Dunford.

If $\sum x_n$ is unconditionally convergent, then the operator $S: X^* \to l$, and hence also S^*, by Schauder's theorem [see (**8**)] defined by

$$S x^* = (x^*(x_1), x^*(x_2), \ldots),$$

is compact, as Dunford has remarked [(**2**) 322; see also Pettis (**7**), Lemma

2.32]. Now choose integers $M \geqslant 1$, $1 \leqslant n_1 < n_2 < \ldots < n_M < \infty$, let $u \equiv (\alpha_1, \alpha_2, \ldots) \in m$ and let f be the element of l^* defined by

$$f(y) \equiv f(\eta_1, \eta_2, \ldots) = \sum_{m=1}^{M} \alpha_{n_m} \eta_{n_m}.$$

Then $\|f\| \leqslant \|u\|$ and† $\qquad S^*f = \sum_{m=1}^{M} \alpha_{n_m} x_{n_m}.$

But, since S^* is strongly compact, this implies that the series $\sum \alpha_n x_n$ satisfies the condition of Theorem 1, and therefore converges unconditionally. If now $h \in l^*$ is defined by

$$h(y) \equiv h(\eta_1, \eta_2, \ldots) = \sum_{n=1}^{\infty} \alpha_n \eta_n,$$

then it is easy to show that

$$S^*h = \sum_{n=1}^{\infty} \alpha_n x_n.$$

T is therefore compact.

Now let c_0 be the subspace of m consisting of all the vectors

$$u \equiv (\alpha_1, \alpha_2, \ldots)$$

such that $\lim_{n\to\infty} \alpha_n = 0$.

COROLLARY. *If X is a weakly complete Banach space, then every bounded operator $T\colon c_0 \to X$ is compact.*

This is a consequence of the elementary observation that the operator has a representation of the form

$$Tu = \sum_{n=1}^{\infty} \alpha_n x_n,$$

where $\sum x_n$ is unconditionally convergent [cf. Gel'fand (**4**)]. It may also be added that, when X is arbitrary, this is the general form for a weakly (and also strongly) compact operator $T\colon c_0 \to X$.

3. Throughout this section it will be supposed that X is a complex Banach space and that $x(\cdot)\colon R \to X$ is a given function of the class $\mathscr{S}$. Since $\mathscr{S} \subseteq \mathscr{D}$, it is possible to write, following Kendall and Moyal,

$$x^*x(t) = \sum_{r=0}^{3} i^r f_r(t; x^*) \quad (x^* \in X^*),$$

where the $f_r(\cdot\,; x^*)$ are real non-decreasing functions. The limits

$$f_r(t-0; x^*)$$

† The x_n are here regarded as embedded, in the canonical sense, in X^{**}.

exist for each t and each x^*, and hence

$$\lim_{h\to 0+} x^*x(t-h) \equiv g(t;x^*)$$

exists. Now suppose that $t_n \uparrow t$ as $n \to \infty$. Then, by Theorem 1, the two series

$$\sum \{x(t_{2r})-x(t_{2r-1})\}, \qquad \sum \{x(t_{2r+1})-x(t_{2r})\}$$

converge unconditionally. Consequently

$$x(t_N) = x(t_1)+\sum_{n=1}^{N-1}\{x(t_{n+1})-x(t_n)\}$$

converge strongly (to x_0, say) as $N \to \infty$. We have

$$x^*(x_0) = g(t;x^*)$$

for all x^* and this shows that x_0 depends on t alone, and not on the choice of the monotonic sequence $\{t_n\}$. Since this sequence was arbitrary, one can deduce without difficulty that

$$\text{strong} \lim_{h\to 0+} x(t-h) = x_0.$$

A similar argument applies to $x(t+0)$.

Next suppose that $u(t) \equiv x(t)-x(t-0)$, that $\{t_n\}$ is an infinite sequence of distinct points in some compact interval $[a,b]$, that $y_n \equiv u(t_n)$, and that $\sum_{m=1}^{\infty} y_{n_m}$ is a subseries of $\sum_{n=1}^{\infty} y_n$. Then

$$\sum_{m=1}^{M} y_{n_m} = \sum_{m=1}^{M}\{x(t_{n_m})-x(t_{n_m}-0)\}$$

$$= \text{strong} \lim_{h\to 0+} \sum_{m=1}^{M}\{x(t_{n_m})-x(t_{n_m}-h)\} \quad (M = 1, 2,...).$$

But for each M a positive $h_0 < 1$ exists such that the intervals

$$[t_{n_m}-h, t_{n_m}] \quad (m = 1, 2,..., M)$$

are disjoint when $0 < h < h_0$. It follows that

$$\sum_{m=1}^{M} x_{n_m} \in \bar{E},$$

where $\bar{E}$ denotes the strong closure of the set E of all finite sums of the form

$$\sum_{k=1}^{K}\{x(s_{2k})-x(s_{2k-1})\}$$

in which $K \geqslant 1$ and $a-1 \leqslant s_1 < s_2 < ... < s_{2K} \leqslant b$. By hypothesis E (and consequently $\bar{E}$) is conditionally weakly compact and therefore,

by Theorem 1, the series $\sum u(t_n)$ converges unconditionally. The method of Kendall and Moyal can now be used to conclude the proof of

THEOREM 3. *If $x(\cdot)$ is a function of Sirvint's class $\mathscr{S}$, then the strong limits $x(t\pm 0)$ exist everywhere, and they satisfy*

$$x(t-0) = x(t) = x(t+0)$$

for all except an (at most) countable set of t-values.

COROLLARY. *Each function of the class $\mathscr{S}$ is separably-valued.*

On writing $\tilde{x}(t) \equiv x(t+0)$ one obtains a new function $\tilde{x}(\cdot)$ which is everywhere strongly continuous to the right and which, as a simple limit argument shows, is also a function of the class $\mathscr{S}$. Suppose this normalization to have been carried out: that is, suppose that the given function $x(\cdot)$ belongs to $\mathscr{S}_0$, the class of functions of $\mathscr{S}$ which are strongly continuous to the right. Let $\{\tau_n\}$ be an enumeration of the discontinuities of $x(\cdot)$. Then we can define $z(\cdot)$ as follows:

$$\left.\begin{aligned} z(t) &= \sum_{0<\tau_n\leqslant t} u(\tau_n) \qquad \text{when } t > 0 \\ z(0) &= 0 \\ z(t) &= -\sum_{t<\tau_n\leqslant 0} u(\tau_n) \quad \text{when } t < 0 \end{aligned}\right\}. \tag{1}$$

If $\{\tau_{n_m}\}$ is any subsequence of $\{\tau_n\}$, then we have, as a consequence of the argument used to prove Theorem 3,

$$\sum_{a<\tau_{n_m}\leqslant b} u(\tau_{n_m}) \in \bar{E},$$

from which result it is easy to deduce that $z(\cdot) \in \mathscr{S}_0$. If $w(t) \equiv x(t)-z(t)$, then it is clear that $w(\cdot)$ is everywhere strongly continuous. This, together with the fact (an easy consequence of Eberlein's theorem) that the difference of two functions in $\mathscr{S}$ is also in $\mathscr{S}$ (so that $w(\cdot) \in \mathscr{S}$) makes it possible to state

THEOREM 4. *If $x(\cdot) \in \mathscr{S}_0$, then there exists a pair of functions $z(\cdot)$, $w(\cdot) \in \mathscr{S}_0$ satisfying*

$$x(t) = z(t)+w(t) \quad (-\infty < t < \infty)$$

and such that $z(\cdot)$ is a step-function, defined at (1), *and $w(\cdot)$ is everywhere strongly continuous.*

4. If $\mathscr{B}$ is any Boolean ring of subsets of R then a function $\mu\colon \mathscr{B} \to X$ will be called a *vector-valued measure* if

$$\mu(\varnothing) = 0, \tag{2}$$

where ø is the empty set, and

$$\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \mu(E_n) \tag{3}$$

whenever the E_n are disjoint sets of $\mathscr{B}$ with their union in $\mathscr{B}$, the series on the right being unconditionally convergent. In what follows $\mathscr{B}$ will denote the class of bounded Borel subsets of R. If $x: R \to X$ is now defined by

$$x(0) = 0, \tag{4}$$

$$x(d) - x(c) = \mu(E) \quad \text{when } E = (c, d), \tag{5}$$

then x is a function in $\mathscr{D}$ and is everywhere strongly continuous to the right. Conversely, if such a function $x \in \mathscr{D}$ is given and if X is weakly complete, then there is a unique vector-valued measure $\mu: \mathscr{B} \to X$ satisfying (5). These results are due, essentially, to Gel'fand (**4**) and they make possible a comparison of the Kendall–Moyal theorems with those of § 3. For, if $\mu: \mathscr{B} \to X$ is a vector-valued measure, then for each $F \in \mathscr{B}$ the set

$$\{\mu(EF): E \in \mathscr{B}\}$$

is, by a theorem of Bartle, Dunford, and Schwartz (**1**), conditionally weakly compact whether X is weakly complete or not. The associated function $x: R \to X$ defined at (4) and (5) is therefore in $\mathscr{S}_0$. So, when X is weakly complete, $\mathscr{S}_0 = \mathscr{D}_0$, where $\mathscr{D}_0$ comprises all the functions of $\mathscr{D}$ which are strongly continuous to the right. Hence

$$\mathscr{S} = \mathscr{D}$$

when X is weakly complete, and it is easy to show similarly that in this case

$$\mathscr{S}(a, b) = \mathscr{D}(a, b) \quad (-\infty < a < b < \infty).$$

The theorems of § 3 may thus be regarded as generalizations of the corresponding Kendall–Moyal theorems.

It will now be shown that the result of Gel'fand's which has just been used can, like the Kendall–Moyal theorems, be extended to arbitrary Banach spaces. We have just seen that, for any given space X and measure $\mu: \mathscr{B} \to X$, the function $x(\cdot)$ defined at (4) and (5) is in $\mathscr{S}_0$. Conversely, it will appear that to any given function $x(\cdot) \in \mathscr{S}_0$ there corresponds a unique measure $\mu: \mathscr{B} \to X$ satisfying (5). Since every function of $\mathscr{S}$ can be normalized, as in § 3, to give a function of $\mathscr{S}_0$, this result will justify the remarks made at the end of § 1.

5. Let $C(a, b)$ denote the Banach space of continuous complex-valued functions $\phi(\cdot)$ on the compact interval $[a, b]$, with

$$\|\phi(\cdot)\| = \sup_{a \leq t \leq b} |\phi(t)|.$$

THEOREM 5 (SIRVINT). *The general form for a weakly compact linear operator* $T: C(a,b) \to X$, *where* X *is an arbitrary complex Banach space, is*

$$T\phi = \int_a^b \phi(t)\, dx(t),$$

where $x(\cdot) \in \mathscr{S}(a,b)$ *and where the integral is a strong Riemann–Stieltjes integral in the sense of Dunford* (**2**).

The original proof of this theorem by Sirvint (**9**) is by no means elementary, and I therefore include the following alternative discussion. The result is closely related to Theorem 3.2 of (**1**) and a part of the proof given below is based on the procedure there employed.

Let $x(\cdot) \in \mathscr{S}(a,b) \subseteq \mathscr{D}(a,b)$. Then [see (**2**)] the strong Riemann–Stieltjes integral

$$T\phi \equiv \int_a^b \phi(t)\, dx(t)$$

exists for each $\phi \in C(a,b)$, and its value is the strong limit of a sequence of sums of the type

$$\sum_{i=1}^{n} \phi(\tau_i)\{x(t_{i+1}) - x(t_i)\},$$

where $$a = t_1 < t_2 < \ldots < t_{n+1} = b$$

and $$t_i \leqslant \tau_i \leqslant t_{i+1} \quad (i = 1, 2, \ldots, n).$$

Now suppose that

$$0 \leqslant \phi(\tau) \leqslant 1 \quad (a \leqslant t \leqslant b).$$

Then the sum $$y \equiv \sum_{i=1}^{n} \phi(\tau_i)\{x(t_{i+1}) - x(t_i)\}$$

can be rewritten as $$\sum_{r=1}^{n} \phi(\tau_{i_r})\{x(t_{i_r+1}) - x(t_{i_r})\}$$

in such a way that

$$0 \leqslant \phi(\tau_{i_1}) \leqslant \phi(\tau_{i_2}) \leqslant \ldots \leqslant \phi(\tau_{i_n}) \leqslant 1.$$

Then

$$y = \phi(\tau_{i_1}) \sum_{r=1}^{n} \{x(t_{i_r+1}) - x(t_{i_r})\} + \sum_{s=2}^{n} \Big[\{\phi(\tau_{i_s}) - \phi(\tau_{i_{s-1}})\} \sum_{r=s}^{n} \{x(t_{i_r+1}) - x(t_{i_r})\}\Big].$$

But this expression makes it evident, in virtue of the inequalities

$$\phi(\tau_{i_1}) \geqslant 0, \qquad \phi(\tau_{i_s}) - \phi(\tau_{i_{s-1}}) \geqslant 0,$$

$$\phi(\tau_{i_1}) + \sum_{s=2}^{n} \{\phi(\tau_{i_s}) - \phi(\tau_{i_{s-1}})\} = \phi(\tau_{i_n}) \leqslant 1,$$

that y, and hence $T\phi$, lies in the closed convex hull K of the set

$$E(a,b) \cup (0).$$

But $E(a,b)$, and therefore $E(a,b) \cup (0)$, is conditionally weakly compact and hence, by a theorem of Krein (**6**), K is weakly compact. Since

$$T\{\phi\colon \|\phi\| \leqslant 1\} \subseteq \sum_{r=0}^{3} i^r x_r\colon x_r \in K$$

it follows, by an application of Eberlein's theorem, that T is a weakly compact operator.

Conversely, if $T\colon C(a,b) \to X$ is weakly compact, then, by a theorem of Bartle, Dunford, and Schwartz (**1**), there exists a vector-valued measure $\mu\colon \mathscr{B} \cap [a,b] \to X$ such that

$$x^*(T\phi) = \int_{[a,b]} \phi(t) x^* \mu\,(dt) \quad (x^* \in X^*). \tag{6}$$

If we now write (in order to avoid an unnecessary use of the Bartle–Dunford–Schwartz integral)

$$y(t) = \mu([a,t]) \quad (a < t \leqslant b), \qquad y(a) = 0, \tag{7}$$

then $y(\cdot) \in \mathscr{S}(a,b)$ and

$$x^*(T\phi) = \int_{[a,b]} \phi(t) x^* \mu\,(dt) = \int_a^b \phi(t)\, dx^*\, y(t)$$

$$= x^*\left\{\int_a^b \phi(t)\, dy(t)\right\} \quad (x^* \in X^*),$$

so that

$$T\phi = \int_a^b \phi(t)\, dy(t)$$

as required.

If now $x(\cdot) \in \mathscr{S}_0$ is given, then the operator $T\colon C(a,b) \to X$ defined by

$$T\phi = \int_a^b \phi(t)\, dx(t)$$

is weakly compact. If the associated $\mu\colon \mathscr{B} \cap [a,b] \to X$ and $y\colon [a,b] \to X$ are constructed as in the foregoing proof [see (6) and (7)], then we have

$$\int_a^b \phi(t)\, dx(t) = \int_a^b \phi(t)\, dy(t) \quad (\phi \in C(a,b)),$$

and then, by a standard argument,

$$x(t) - x(a) = y(t) - y(a) \quad (a \leqslant t \leqslant b).$$

Hence $y(a) = y(a+0)$ and therefore $\mu((a)) = 0$, so that

$$y(t) = \mu([a,t]) = \mu([a,t]) - \mu((a)) = \mu((a,t]) \quad (a < t \leqslant b).$$

If we now denote by μ_0 the restriction of μ to $\mathscr{B} \cap (a, b]$, then we have

$$\mu_0(E) = y(d) - y(c) = x(d) - x(c)$$

whenever $E = (c, d] \subseteq (a, b]$.

We can so construct μ_0 whenever $-\infty < a < b < \infty$, and it is now possible, by a simple argument, to complete the proof of the following

COROLLARY. *If $x(\cdot)\colon R \to X$ is a function of the class $\mathscr{S}_0$, where X is an arbitrary Banach space, then there exists a unique vector-valued measure $\mu\colon \mathscr{B} \to X$ satisfying* (5).

This justifies the assertion made at the end of § 4.

In conclusion, I thank Mr. D. G. Kendall and Mr. J. E. Moyal for communicating their theorems to me before publication; I am also particularly indebted to Mr. Kendall for a number of suggestions which have materially improved the present paper.

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INTEGRAL TRANSFORMS OVER CERTAIN FUNCTION SPACES (I)

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1. Introduction

THE functions f, F in $L^2(0,\infty)$ are transforms of one another if they satisfy a pair of relations of the type

$$\int_0^w F(t)\,dt = \int_0^\infty f(x)\chi(x,w)\,dx, \tag{1.1}$$

$$\int_0^x f(y)\,dy = \int_0^\infty F(t)\rho(x,t)\,dt, \tag{1.2}$$

where χ, ρ satisfy appropriate conditions. The relations

$$\int_0^w F(t)\,dt = \int_0^\infty f(x)\,dx \int_0^w \phi(x,t)\,dt, \tag{1.3}$$

$$\int_0^x f(y)\,dy = \int_0^\infty F(t)\,dt \int_0^x \phi(y,t)\,dy \tag{1.4}$$

were considered first by Temple (**5**), and subsequently by a number of writers. [Various references will be found in (**2**).] As far as the space $L^2(0,\infty)$ is concerned, with Lebesgue integrals, the group of theorems given by Bochner and Chandrasekharan (**1**) solves completely the problem of the characterization of unitary transformations by 'kernels' χ and ρ. They remark also that more general situations can be dealt with by their methods. One such generalization arises in considering the 'spectral theorem' for ordinary linear differential equations of second order with one singular point, where we are concerned with a one-to-one linear mapping of $L^2(0,\infty)$ on to a space generated by a Lebesgue–Stieltjes measure, also of L^2 type [see (**4**)].

Given two spaces of L^2 type, not necessarily the same, I shall call a transformation from one to the other *unitary* if it is a one-to-one linear mapping which preserves inner products. In the present paper I extend the results of (**1**) (Theorems 77, 78) to apply to L^2 spaces associated with abstract measure spaces, and to spaces whose elements are in other L^2

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spaces. Some additional problems will be considered in a subsequent paper.

In the terminology of Halmos (**3**), one can extend the results of (**1**) simply by replacing characteristic functions over intervals in $(0, \infty)$ by characteristic functions of sets of finite measure over a σ-ring. The results obtained in this way are not, however, a true generalization of the theorems of Bochner and Chandrasekharan, which assert that certain consequences follow if χ and ρ satisfy conditions over a ring included in the σ-ring of all Lebesgue-measurable sets in $(0, \infty)$. I begin, therefore, by introducing the appropriate 'auxiliary ring' into the abstract space. The application to the Euclidean case is considered in § 5.

2. An auxiliary ring of sets

Let (δ, s_0, μ) be a measure space, i.e. δ is a set, s_0 a σ-ring of subsets of δ, μ a measure. We suppose that δ is σ-finite, and μ non-negative. The function space l^2 (elements f, g,...) is then the class of measurable functions f with values in the field of complex numbers, for which $|f|^2$ is integrable over δ, and

$$\|f\| = \left\{\int_\delta |f|^2 \, d\mu\right\}^{\frac{1}{2}}, \qquad (f,g) = \int_\delta f\bar{g} \, d\mu,$$

where $\bar{g}$ denotes the conjugate complex of g.

Let s be a subclass of s_0, with elements denoted by p, q,..., with the following properties:

(i) all sets of s are of finite measure μ;

(ii) all finite disjoint unions of sets of s are in s;

(iii) if $p, q \in s$, then $p-q \in s$;

(iv) if $p_1,..., p_n$ are disjoint sets of s, $\mathrm{c}p_1,..., \mathrm{c}p_n$ their characteristic functions, $a_1,..., a_n$ arbitrary complex numbers, then the class of all simple functions $\sum a_r \mathrm{c}p_r$ is dense in l^2.

Clearly finite intersections of sets in s are also sets in s. Every measure space must contain such a class, for the class of sets in s_0 all of which are of finite measure satisfies the above conditions.

If $p, q \in s$, and a, b are complex numbers, then

$$a\mathrm{c}p+b\mathrm{c}q = (a+b)\mathrm{c}(p \cap q)+a\mathrm{c}(p-q)+b\mathrm{c}(q-p). \qquad (2.1)$$

Thus s is a ring of sets in δ over which the simple functions constitute a linear manifold dense in l^2.

The following properties of the class s will be required.

LEMMA 1. *If $f \in l^2$, there exist numbers a_i and a finite number of disjoint sets $p_i \in s$, for which*

(i) $$\left| \int_\delta |f|^2 \, d\mu - \sum a_i \int_{p_i} \bar{f} \, d\mu \right| < \epsilon,$$

(ii) $$\left| \int_\delta |f|^2 \, d\mu - \sum |a_i|^2 \mu(p_i) \right| < \epsilon.$$

By hypothesis, there exists a sequence of simple functions $\sum a_{in} \, \mathrm{c}p_{in}$ converging to f in l^2 norm, so that, as $n \to \infty$,

$$\int_\delta \bar{f}(f - \sum a_{in} \, \mathrm{c}p_{in}) \, d\mu \to 0,$$

$$\int_\delta \left| \sum a_{in} \, \mathrm{c}p_{in} \right|^2 d\mu \to \int_\delta |f|^2 \, d\mu.$$

Remembering that, for each n, the sets p_{in} are disjoint, we see that the results stated follow immediately.

LEMMA 2. *If $f \in l^2$ and $\int_p f \, d\mu = 0$ for every $p \in s$, then $f = 0$.*

This is an immediate consequence of (i) of Lemma 1.

3. The Temple–Bochner theory

We shall be concerned with unitary transformations (as previously defined) between a pair of function spaces of the type considered in § 2. In order to simplify the notation, as far as possible lower-case letters will refer to one space, and corresponding upper-case letters to the other. More precisely, the measure space (δ, s_0, μ) determines some class $s \subset s_0$, with elements $p, q, \ldots$, and a function space l^2, with elements $f, g, \ldots$. We are given also a measure space (Δ, S_0, ν) of similar type, a ring $S \subset S_0$ with elements $P, Q, \ldots$, and the space determines a function space L^2 with elements $F, G, \ldots$. Point functions over δ, Δ will be written $f(x)$, $F(y)$ respectively.

A function $\gamma(P)$, or $\gamma(P, x)$, which is both a function of a set P and a point function, will be called a $\mu\nu S$-kernel if it satisfies the following set of conditions:

(i) $\gamma(P, x)$ is defined for every $P \in S$, and for almost all x in δ, the 'almost all' referring to the measure μ;

(ii) $\gamma(P) \in l^2$ for every $P \in S$;

(iii) $\gamma(P)$ is finitely additive on S;

(iv) $\gamma(P)$ is the zero element of l^2 whenever $\nu(P) = 0$.

Similarly, a $\nu\mu s$-kernel $\Gamma(p,y) = \Gamma(p)$ is defined for all $p \in s$, for almost all y in Δ (with respect to ν), is in L^2 for every $p \in s$, finitely additive on s, and is the zero element of L^2 if $\mu(p) = 0$.

THEOREM 1. *To each unitary transformation T from l^2 to L^2 correspond a unique $\mu\nu S$-kernel γ and a unique $\nu\mu s$-kernel Γ which satisfy the following relations, for all $p, q \in s$, $P, Q \in S$:*

$$(A) \quad \int_p \bar{\gamma}(P)\, d\mu = \int_P \Gamma(p)\, d\nu,$$

$$(K_1) \quad \int_\delta \gamma(P)\bar{\gamma}(Q)\, d\mu = \nu(P \cap Q),$$

$$(K_2) \quad \int_\Delta \Gamma(p)\overline{\Gamma}(q)\, d\nu = \mu(p \cap q).$$

If $Tf = F$, $T^{-1}F = f$, then

$$(T_1) \quad \int_P F\, d\nu = \int_\delta f\bar{\gamma}(P)\, d\mu,$$

$$(T_2) \quad \int_p f\, d\mu = \int_\Delta F\,\overline{\Gamma}(p)\, d\nu,$$

again for all $p \in s$, $P \in S$.

Proceeding as in (**1**), we define $\Gamma(p) = T\mathfrak{c}p$, $\gamma(P) = T^{-1}\mathfrak{c}P$, for all $p \in s$, $P \in S$ respectively. Since T and T^{-1} preserve inner products, the relations (K_1)–(T_2) follow immediately, and the condition of adjointness (A) can be verified by taking $f = \mathfrak{c}p$ in (T_1). In order to show that the kernels are unique, suppose, for example, that γ' is the difference of two $\mu\nu S$-kernels with the properties of γ. Then, by (T_1), the inner product (f, γ') vanishes for every $P \in S$, and for every $f \in l^2$. It then follows from Lemma 2 that γ' is the zero element of l^2, for every P.

THEOREM 2. *Suppose given a $\mu\nu S$-kernel γ and a $\nu\mu s$-kernel Γ for which (A), (K_1), (K_2) hold over s and S. Then (T_1), (T_2) hold, and the transformations $Tf = F$, $T'F = f$ constitute a unitary transformation T from l^2 to L^2 and its inverse.*

Throughout the proof, simple functions are defined over finite unions of disjoint sets of s or S.

We define a transformation T_0 from l^2 to L^2 with domain these simple functions of l^2, by the relation

$$T_0 f = \sum_{r=1}^{n} a_r \Gamma(p_r) \quad \text{when } f = \sum_{r=1}^{n} a_r \mathfrak{c}p_r.$$

Suppose that any other representation of f is

$$f = \sum_{s=1}^{m} a'_s \mathrm{c}p'_s, \qquad \bigcup p_r = \bigcup p_s$$

except possibly for a null set. Since Γ is finitely additive and vanishes on null sets,

$$\sum_{r=1}^{n} a_r \Gamma(p_r) = \sum_{r=1}^{n} a_r \sum_{s=1}^{m} \Gamma(p_r \cap p'_s) = \sum_{r=1}^{n} a_r \sum_{a_r = a'_s} \Gamma(p_r \cap p'_s)$$

$$= \sum_{s=1}^{n} a'_s \sum_{r=1}^{n} \Gamma(p_r \cap p'_s) = \sum_{s=1}^{n} a'_s \Gamma(p'_s).$$

For a pair of simple functions

$$f = \sum_{r=1}^{n} a_r \mathrm{c}p_r, \qquad g = \sum_{s=1}^{m} b_s \mathrm{c}q_s,$$

$$(T_0 f, T_0 g) = \sum\sum a_r \bar{b}_s (\Gamma(p_r), \Gamma(q_s)) = \sum\sum a_r \bar{b}_s \mu(p_r \cap q_s)$$

by (K_2), so that $(T_0 f, T_0 g) = (f, g)$. Next, if α, β are complex numbers,

$$p = \bigcup p_r, \qquad q = \bigcup q_s,$$

by (2.1),

$$\begin{aligned} T_0(\alpha f + \beta g) &= T_0\Big\{ \sum_{r,s} (\alpha a_r + \beta b_s)\mathrm{c}(p_r \cap q_s) + \sum_r \alpha a_r \,\mathrm{c}(p_r - q) + \sum_s \beta b_s \,\mathrm{c}(q_s - p)\Big\} \\ &= \sum_{r,s} (\alpha a_r + \beta b_s)\Gamma(p_r \cap q_s) + \sum_r \alpha a_r \,\Gamma(p_r - q) + \sum_s \beta b_s \,\Gamma(q_s - p) \\ &= \alpha \sum_r a_r \{\Gamma(p_r \cap q) + \Gamma(p_r - q)\} + \beta \sum_s b_s \{\Gamma(p \cap q_s) + \Gamma(q_s - p)\} \\ &= \alpha \sum_r a_r \Gamma(p_r) + \beta \sum_s b_s \Gamma(q_s) \\ &= \alpha T_0 f + \beta T_0 g. \end{aligned}$$

It has now been shown that T_0 is a bounded linear operator preserving inner products, with domain dense in l^2. It follows that its linear extension T (say) is also bounded and linear, preserves inner products, and that its domain is l^2. By using (K_1) in place of (K_2), we obtain a second operator T' with these properties, but with domain L^2. By (A),

$$(\mathrm{c}p, T'\mathrm{c}P) = (T\mathrm{c}p, \mathrm{c}P)$$

holds for every $p \in s$, $P \in S$. Hence

$$(f, T'G) = (Tf, G) \tag{3.1}$$

holds, first for characteristic functions, then for simple functions, and finally, by the continuity of inner products, for all $f \in l^2$, $G \in L^2$. Thus T, T' are adjoint.

Finally, let $G, H \in L^2$, and let $T'G = g$. Then, by (3.1),

$$(Tg, H) = (TT'G, H) = (T'G, T'H) = (G, H)$$

holds for every $H \in L^2$, i.e. $Tg = G$. There is a similar argument for T'. Thus the range of T is L^2, the range of T' is l^2, T, T' are inverse, and T is unitary.

4. Absolutely continuous kernels

Formulae of the type (1.3), (1.4) arise when the kernels are absolutely continuous. Actually the term kernel is more appropriately reserved for functions $\phi(x,t)$, here called a *Fourier kernel*. The corresponding theorem given by Bochner and Chandrasekharan extends quite simply to the present situation. In this connexion, the Cartesian product of δ, Δ will be denoted by $\delta \times \Delta$, and similarly for sets.

THEOREM 3. *Suppose that the conditions of Theorem 2 are satisfied by kernels γ, Γ, that there exists a Fourier kernel $\phi = \phi(x,y)$ for which*

$$\int_{p \times P} |\phi|^2 \, d\mu dv < \infty \tag{4.1}$$

for every $p \in s$, $P \in S$, and that

$$\gamma(P,x) = \int_P \bar{\phi}(x,y) \, dv(y) \tag{4.2}$$

for every $P \in S$. Then, for every $p \in s$,

$$\Gamma(p,y) = \int_p \phi(x,y) \, d\mu(x). \tag{4.3}$$

Let $\{p_n\}$ be any ascending sequence of sets of s converging to δ, $\{P_n\}$ any ascending sequence of sets of S converging to Δ. Let

$$F_n = \int_{p_n} f(x)\phi(x,y) \, d\mu(x) \quad (f \in l^2),$$

$$f_n = \int_{P_n} F(y)\bar{\phi}(x,y) \, dv(y) \quad (F \in L^2).$$

Then, as $n \to \infty$, $F_n \to Tf$ in L^2 norm, and $f_n \to T^{-1}F$ in l^2 norm.

Let

$$h_n(x) = \begin{cases} f(x) & (x \in p_n), \\ 0 & \text{otherwise} \end{cases}$$

Then $\|f-h_n\| \to 0$. For any $P \in S$,

$$\int_{\dot{P}} F_n \, d\nu = \int_{\dot{P}} d\nu(y) \int_{p_n} f(x)\phi(x,y) \, d\mu(x)$$

$$= \int_{p_n} f(x)\bar{\gamma}(P,x) \, d\mu(x)$$

$$= \int_{\delta} h_n(x)\bar{\gamma}(P,x) \, d\mu(x).$$

The inversion is justified by using the Schwarz inequality and the general Fubini theorem. Thus

$$\left\{\int_{\dot{P}} d\nu(y) \int_{p_n} |f(x)\phi(x,y)| \, d\mu(x)\right\}^2$$

$$\leqslant \nu(P) \int_{\dot{P}} \left\{\int_{p_n} |f(x)\phi(x,y)| \, d\mu(x)\right\}^2 d\nu(y)$$

$$\leqslant \nu(P) \int_{\dot{P}} d\nu(y) \int_{p_n} |f|^2 \, d\mu \int_{p_n} |\phi(x,y)|^2 \, d\mu(x)$$

$$\leqslant \nu(P)\|f\|^2 \int_{p_n \times P} |\phi|^2 \, d\mu d\nu < \infty,$$

by (4.1). It now follows by (T_1) of Theorem 1 that $Th_n = F_n$, and

$$\|Tf-F_n\| = \|Tf-Th_n\| = \|f-h_n\| \to 0.$$

By (4.2),

$$\int_{\dot{p}} \bar{\gamma}(P) \, d\mu = \int_{\dot{p}} d\mu(x) \int_{\dot{P}} \phi(x,y) \, d\nu(y)$$

$$= \int_{\dot{P}} d\nu(y) \int_{\dot{p}} \phi(x,y) \, d\mu(x),$$

the inversion being justified as before. By (A) of Theorem 1, and the uniqueness of the kernel, we have (4.3).

5. Euclidean spaces

If one or both of the spaces δ, Δ is Euclidean, and one or both of the measures is a Stieltjes–Lebesgue measure, the conditions take on the simpler form given by Temple and Bochner. In such cases the class s, for example, may be chosen to be the set of all finite disjoint unions of half-open intervals in the space. Further simplification is possible, as is shown in the special case formulated below.

THEOREM 4. *Suppose that δ is $(0,\infty)$, μ a Stieltjes–Lebesgue measure, and let Γ_a denote the set function $\Gamma(p)$ with p the interval $0 \leqslant x < a$. Then the conditions (A) and (K_2) of Theorem 1 may be replaced by*

$$\int_0^a \bar{\gamma}(P)\, d\mu = \int_P \Gamma_a\, d\nu, \tag{5.1}$$

$$\int_\Delta \Gamma_a \overline{\Gamma}_b\, d\nu = \mu\{(0,a) \cap (0,b)\}, \tag{5.2}$$

the first of which must hold for all $a > 0$, $P \in S$, and the second for all $a, b > 0$.

Suppose, for example, that (5.2) holds, let Γ_{ab}, Γ_{cd} denote $\Gamma(p)$ with p equal to (a,b), (c,d), and let μ_a denote the measure of $(0,a)$. If $0 < a < c < b < d$, since Γ is additive, we have

$$\begin{aligned}\int_\Delta \Gamma_{ab}\, \overline{\Gamma}_{cd}\, d\nu &= \int_\Delta (\Gamma_b - \Gamma_a)(\overline{\Gamma}_d - \overline{\Gamma}_c)\, d\nu \\ &= \mu_b - \mu_c \\ &= \mu\{(a,b) \cap (c,d)\},\end{aligned} \tag{5.3}$$

and similarly in all other cases. If now $p = \bigcup I_r$, $q = \bigcup J_r$ are finite disjoint sets of intervals in s, we have

$$\begin{aligned}\int_\Delta \Gamma(p)\overline{\Gamma}(q)\, d\nu &= \int_\Delta \{\sum \Gamma(I_r)\}\{\sum \overline{\Gamma}(J_s)\}\, d\nu \\ &= \sum\sum \int_\Delta \Gamma(I_r)\overline{\Gamma}(J_s)\, d\nu \\ &= \sum\sum \mu(I_r \cap J_s),\end{aligned}$$

by (5.3). Now, for fixed r, the sets $I_r \cap J_s$ are disjoint, so the sum is equal to

$$\sum_r \mu(I_r \cap \{\bigcup J_s\}) = \mu(\{\bigcup I_r\} \cap \{\bigcup J_s\}) = \mu(p \cap q).$$

The other result follows in a similar manner.

Suppose, for example, that we wish to derive Plancherel's theorem for Fourier cosine-transforms from Theorem 3. Then

$$\phi(x,y) = \sqrt{\left(\frac{2}{\pi}\right)} \cos xy,$$

and we need only verify that

$$\frac{2}{\pi}\int_0^\infty \frac{\sin ax \sin bx}{x^2}\, dx = \min(a,b).$$

Clearly, the use of rings s, S is necessary in order that the results obtained may be used in particular cases.

As another example in Euclidean space, suppose that ϕ satisfies the conditions of Theorem 3, that Δ is the real line, and ν the measure obtained by integrating with respect to a non-decreasing step-function with jumps of unity at (y_r), where $-\infty < y_0 < y_1 < \ldots$. Let

$$\phi(x, y_r) = \psi_r(x).$$

The kernels are now

$$\gamma(P, x) = \sum_{y_r \in P} \psi_r(x), \qquad \Gamma(p, y_r) = \int_p \psi_r(x)\, d\mu.$$

Evidently the results of Theorem 3 now reduce to elementary facts about orthonormal sequences.

6. Spaces with elements in an l^2 space

Results analogous to those of §§ 3, 4 hold also in a more general setting. Suppose given a measurable space (δ, s_0), so that δ is the union of all the sets of s_0, and a sequence $\mu_1, \mu_2, \ldots$, of non-negative measures, associated with the same classes s_0, s. Corresponding to each measure μ_i will be a function space $l^2(\mu_i)$. Let λ denote the space of elements $f, g, \ldots$ consisting of all ordered sequences

$$f = [f_1, f_2, \ldots], \qquad g = [g_1, g_2, \ldots], \ldots$$

where f_i, g_i are elements of $l^2(\mu_i)$. (Similar results are obtained if we fix an integer n and construct an 'n-dimensional' space in the same way.) We now define $f = g$ when $f_i = g_i$ (all i), $af = [af_1, af_2, \ldots]$ for all complex numbers a, and $f+g = [f_1+g_1, f_2+g_2, \ldots]$, so that λ is a linear space.

Let e be any set in s_0, and let $\mathrm{c}_{ij}e$ denote the characteristic function of e, regarded as an element of $l^2(\mu_i)$, when $j = i$, and the zero element of $l^2(\mu_j)$ when $j \neq i$. In this way we obtain a sequence of elements of λ,

$$\mathrm{c}_i e = [\mathrm{c}_{i1}e, \mathrm{c}_{i2}e, \ldots, \mathrm{c}_{ii}e, \ldots], \tag{6.1}$$

which will be referred to as the 'characteristic function' of e in λ. An element f of λ will be called *simple* when f_i is a simple function in $l^2(\mu_i)$ for every value of i.

We could now define an inner product in λ by the relation

$$(f, g) = \sum_{i=1}^{\infty} (f_i, g_i),$$

and then λ would be the direct product of the spaces $l^2(\mu_i)$. In fact, we proceed more generally as follows.

We suppose that an inner product $\{f, g\}$ has been defined in λ, and that λ is a complete inner-product space in which the simple elements over s are dense.

Consider now two systems satisfying these conditions, specified by the symbols

$$(\delta, s_0, \mu_1, \mu_2, \ldots), \quad s \text{ (sets } p, q, \ldots), \quad \lambda \text{ (elements } f, g, \ldots),$$
$$(\Delta, S_0, \nu_1, \nu_2, \ldots), \quad S \text{ (sets } P, Q, \ldots), \quad \Lambda \text{ (elements } F, G, \ldots).$$

No confusion will arise between λ, Λ if their inner products are denoted by the same symbol.

A $\mu\nu S$-kernel γ can be defined as follows. The kernel is a sequence of point-set functions $\gamma^{(i)}(P)$ satisfying the following conditions (if, however, Λ has 'dimension' N, then $1 \leqslant i \leqslant N$: this means that Λ is constructed from sets of N elements):

(i) $\gamma^{(i)}(P)$ is defined over all $P \in S$, all i;

(ii) $\gamma^{(i)}(P) \in \lambda$, so that we may write

$$\gamma^{(i)}(P) = [\gamma_1^{(i)}(P), \gamma_2^{(i)}(P), \ldots], \qquad \gamma_j^{(i)}(P) \in l^2(\mu_j);$$

(iii) for each i, $\gamma^{(i)}(P)$ is finitely additive over S (this is equivalent to the condition that, for each i, j, $\gamma_j^{(i)}$ be finitely additive over S);

(iv) $\gamma^{(i)}(P)$ is the zero element of λ whenever $\nu_i(P) = 0$.

Similarly we define a $\nu\mu s$-kernel Γ by interchange of (s, λ, μ_i), (S, Λ, ν_i). If λ is of dimension n, then Γ consists of n elements of Λ.

THEOREM 5. *To each unitary transformation T from λ to Λ correspond a unique $\mu\nu S$-kernel γ and a unique $\nu\mu s$-kernel Γ, satisfying the relations*

$$(A) \qquad \{\mathrm{c}_i p, \gamma^{(j)}(P)\} = \{\Gamma^{(i)}(p), \mathrm{c}_j(P)\},$$
$$(K_1) \quad \{\gamma^{(i)}(P), \gamma^{(j)}(Q)\} = \{\mathrm{c}_i P, \mathrm{c}_j Q\},$$
$$(K_2) \quad \{\Gamma^{(i)}(p), \Gamma^{(j)}(q)\} = \{\mathrm{c}_i p, \mathrm{c}_j q\},$$

for all $i, j, p \in s, P \in S$.

If $Tf = F$, $T^{-1}F = f$, then

$$(T_1) \quad \{F, \mathrm{c}_j P\} = \{f, \gamma^{(j)}(P)\}, \qquad (T_2) \quad \{f, \mathrm{c}_i p\} = \{F, \Gamma^{(i)}(p)\}$$

for all $i, j, p \in s, P \in S$.

Conversely, if γ, Γ are $\mu\nu S$, $\nu\mu s$-kernels respectively for which (A), (K_1), (K_2) hold, then (T_1), (T_2) provide a unitary transformation T from λ to Λ, with $Tf = F$, and its inverse.

To prove the first part, we define, for each i, j, p, P,

$$\Gamma^{(i)}(p) = T\mathrm{c}_i p, \qquad \gamma^{(j)}(P) = T^{-1}\mathrm{c}_j(P),$$

and proceed as in § 3. For the converse, let

$$f = \sum_{\alpha=1}^{\infty} \sum_{r=1}^{\rho} a_{r\alpha}\, c_\alpha\, p_{r\alpha} \tag{6.2}$$

be any simple element of λ, in which $p_{r\alpha}$ are disjoint sets of s for

$$1 \leqslant r \leqslant \rho = \rho(\alpha),$$

and $a_{r\alpha}$ are complex numbers. We then define a transformation T_0 from λ to Λ over these simple elements by the relation

$$T_0 f = \sum_{\alpha=1}^{\infty} \sum_{r=1}^{\rho} a_{r\alpha}\, \Gamma^{(\alpha)}(p_{r\alpha}).$$

It must be shown in the first instance that $T_0 f$ exists, and that T_0 is single-valued. If

$$f^{(N)} = \sum_{\alpha=1}^{N} \sum_{r=1}^{\rho} a_{r\alpha}\, c_\alpha\, p_{r\alpha}, \qquad F^{(N)} = \sum_{\alpha=1}^{N} \sum_{r=1}^{\rho} a_{r\alpha}\, \Gamma^{(\alpha)}(p_{r\alpha}),$$

then, using (K_2), we have

$$\begin{aligned} \|F^{(N)}\|^2 &= \sum_{\alpha\beta} \sum_{rs} a_{r\alpha}\, \bar{a}_{s\beta} \{\Gamma^{(\alpha)}(p_{r\alpha}), \Gamma^{(\beta)}(p_{s\beta})\} \\ &= \sum_{\alpha\beta} \sum_{rs} a_{r\alpha}\, \bar{a}_{s\beta} \{c_\alpha p_{r\alpha}, c_\beta p_{s\beta}\} = \|f^{(N)}\|^2. \end{aligned}$$

It follows that $T_0 f$ exists and $\|T_0 f\| = \|f\|$. Next, let g be the simple element of λ given by

$$g = \sum_{\alpha=1}^{\infty} \sum_{r=1}^{\sigma} b_{r\alpha}\, c_\alpha\, q_{r\alpha}, \qquad T_0 g = \sum_{\alpha=1}^{\infty} \sum_{r=1}^{\sigma} b_{r\alpha}\, \Gamma^{(\alpha)}(q_{r\alpha}),$$

where the sets $q_{r\alpha} \in s$, and are disjoint for $1 \leqslant r \leqslant \sigma = \sigma(r)$. If $f = g$, where f is given by (6.2), then

$$\bigcup_r p_{r\alpha} = \Big(\bigcup_r q_{r\alpha}\Big) \cup p, \qquad \mu_\alpha(p) = 0, \qquad p = p_\alpha.$$

Hence

$$\begin{aligned} \sum_{r=1}^{\rho} a_{r\alpha}\, \Gamma^{(\alpha)}(p_{r\alpha}) &= \sum_{r=1}^{\rho} a_{r\alpha} \sum_{s=1}^{\sigma} \Gamma^{(\alpha)}(p_{r\alpha} \cap q_{s\alpha}) \\ &= \sum_{s=1}^{\sigma} b_{s\alpha} \sum_{r=1}^{\rho} \Gamma^{(\alpha)}(p_{r\alpha} \cap q_{s\alpha}) = \sum_{s=1}^{\sigma} b_{s\alpha}\, \Gamma^{(\alpha)}(q_{s\alpha}), \end{aligned}$$

precisely as in the proof of Theorem 2. Hence $T_0 f = T_0 g$, and T_0 is single-valued.

We prove similarly that T_0 is linear. As before, we obtain transformations T, T' with domains λ, Λ respectively. Suppose next that f is given by (6.2) and that G is any simple element of Λ, say

$$G = \sum_{\beta=1}^{\infty} \sum_{s=1}^{\sigma} A_{s\beta}\, c_\beta\, P_{s\beta}, \qquad T'G = \sum_{\beta=1}^{\infty} \sum_{s=1}^{\sigma} A_{s\beta}\, \gamma^{(\beta)}(P_{s\beta}),$$

where $A_{s\beta}$ are complex numbers, and $P_{s\beta} \in S$ and are disjoint for

$$1 \leqslant s \leqslant \sigma = \sigma(\beta).$$

Then, using (A), we have

$$\begin{aligned}\{f, T'G\} &= \sum_{\alpha\beta rs} a_{r\alpha}\bar{A}_{s\beta}\{c_\alpha p_{r\alpha}, \gamma^{(\beta)}(P_{s\beta})\} \\ &= \sum_{\alpha\beta rs} a_{r\alpha}\bar{A}_{s\beta}\{\Gamma^{(\alpha)}(p_{r\alpha}), c_\beta P_{s\beta}\} \\ &= \{Tf, G\}.\end{aligned}$$

Thus the analogue of (3.1) holds for simple elements, and the proof is completed as before.

It should be noted that the measures μ_i, ν_i appear explicitly in conditions (iv) for the kernels only, but that the sets s (and S) must be the same for all μ_i (and ν_i).

The theory can be further generalized by considering isometric transformations between closed linear manifolds in λ and Λ. This can be done by replacing the characteristic function of (6.1) by a set of elements of λ whose 'components' are arbitrary simple elements.

7. Direct product spaces

We consider now the modifications of the preceding results which arise if, in λ,

$$\{f, g\} = \sum_{i=1}^{\infty} (f_i, g_i), \qquad \|f\|^2 = \sum_{i=1}^{\infty} \|f_i\|^2 = \sum_{i=1}^{\infty} \int_\delta |f_i|^2 \, d\mu_i,$$

and, in Λ,

$$\{F, G\} = \sum_{i=1}^{\infty} (F_i, G_i), \qquad \|F\|^2 = \sum_{i=1}^{\infty} \|F_i\|^2 = \sum_{i=1}^{\infty} \int_\Delta |F_i|^2 \, d\nu_i.$$

Then λ is the direct product of $l^2(\mu_1)$, $l^2(\mu_2)$,..., which is known to be complete, and similarly for Λ. Granted suitable rings of sets s, S, the results of Theorem 5 hold with the following system of equations, in which $1 \leqslant i, j \leqslant \infty$, and δ_{ij} is the Kronecker delta:

$$(A) \quad \int_p \bar{\gamma}_i^{(j)}(P) \, d\mu_i = \int_P \Gamma_j^{(i)}(p) \, d\nu_j,$$

$$(K_1) \quad \sum_{\alpha=1}^{\infty} \int_\delta \gamma_\alpha^{(i)}(P)\bar{\gamma}_\alpha^{(j)}(Q) \, d\mu_\alpha = \delta_{ij}\nu_i(P \cap Q),$$

$$(K_2) \quad \sum_{\beta=1}^{\infty} \int_\Delta \Gamma_\beta^{(i)}(p)\bar{\Gamma}_\beta^{(j)}(q) \, d\nu_\beta = \delta_{ij}\mu_i(p \cap q),$$

$$(T_1) \quad \int_P F_j \, d\nu_j = \sum_{\alpha=1}^{\infty} \int_\delta f_\alpha \bar{\gamma}_\alpha^{(j)}(P) \, d\mu_\alpha,$$

$$(T_2) \quad \int_p f_i \, d\mu_i = \sum_{\beta=1}^{\infty} \int_\Delta F_\beta \bar{\Gamma}_\beta^{(i)}(p) \, d\nu_\beta.$$

If, however, the 'dimensions' of λ, Λ are n, N respectively, the integers i, j have the following ranges:

$$(A) \quad (1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant N), \qquad (K_1) \quad (1 \leqslant i, j \leqslant N),$$

$$(K_2) \quad (1 \leqslant i, j \leqslant n), \qquad (T_1) \quad (1 \leqslant j \leqslant N), \qquad (T_2) \quad (1 \leqslant i \leqslant n).$$

In this case it is easily seen that, if the simple functions over s are dense in each space $l^2(\mu_i)$, then the simple elements of λ are dense in λ.

8. Partitioned spaces

An apparently different case arrives when we have a measurable space (δ, s_0), and partition it into a finite or infinite sequence of measurable sets (say the latter) which are non-empty and disjoint, say $\delta = \bigcup \delta_i$. Suppose that s_{0i} is the class of sets defined by the condition

$$e \in s_{0i} \quad \text{if there exists } E \in s_0 \text{ for which } e = E \cap \delta_i.$$

Then, for each i, s_{0i} is a σ-ring over δ_i, and, if we define a measure μ_i' in each (δ_i, s_{0i}), so that $(\delta_i, s_{0i}, \mu_i')$ is a measure space of the required type, we may construct a function space, say l_i^2, over $(\delta_i s_{0i}, \mu_i')$. In this way we obtain a sequence of l^2 spaces as before. Finally we can construct a linear space λ' as the class of elements $[f_1, f_2, \ldots]$ with $f_i \in l_i^2$, and obtain a result corresponding to Theorem 5.

It turns out, however, that every λ' is a subspace of some λ. For we can define a sequence of measures μ_i over (δ, s_0) by the relations

$$\mu_i(e) = \mu_i'(e \cap \delta_i),$$

and the situation is reduced to that of § 6.

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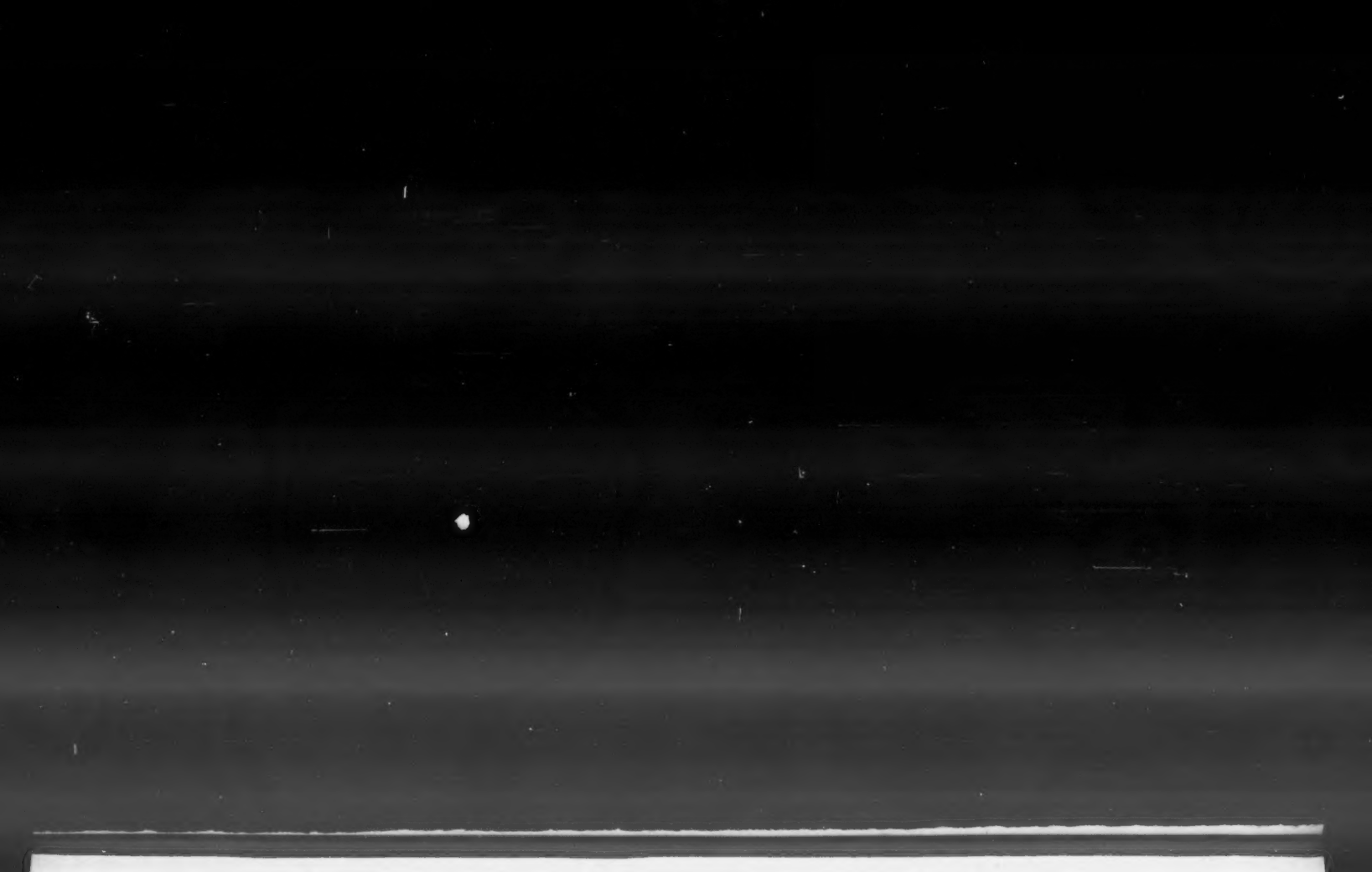
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